Stable Reduction of $X_0(625)$ over $\mathbb{C}_5$
(with implications)

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$X_0(p^n)$ is the smooth, projective curve over $\mathbb{Q}_p$ whose $\mathbb{C}_p$-valued points correspond to pairs $(E, C)$, where $E$ is an elliptic curve and $C \subseteq E$ is a cyclic subgroup of order $p^n$. 
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Every curve over $\mathbb{Q}_p$ has a model over $\mathcal{O}_K$, for some finite extension $K$ of $\mathbb{Q}_p$, whose special fiber has at worst ordinary double points (i.e., $\hat{\mathcal{O}}_{\mathcal{C}, \mathcal{P}} \cong K[[x, y]]/(xy)$).
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**Goal:** Compute the stable reduction of $X_0(p^n)$.
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Rigid-Analytic Reformulation

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X_0(p^n) = \left( \bigcup_{a+b=n} (X^+_a \cup X^-_a) \right) \cup \left( \bigcup_A W_A(p^n) \right).
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$X_{a,b}^+ \cup X_{a,b}^-$ contains those pairs $(E, C)$ where $E$ has ordinary reduction, $K(E)$ is the kernel of reduction, and $|K(E) \cap C| = p^a$. 

Question: So why aren't we done?

Answer: As $n$ grows, the reduction of $W_A(p^n)$ contains more components.
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A Good Starting Point

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Description of $X_0(p^n)$ for $n \leq 3$

$\mathcal{W}_A(\rho)$ is an annulus. $X_{10}$ and $X_{01}$ are copies of the (ordinary) $j$-line. So the stable reduction of $X_0(\rho)$ just consists of two copies of $\mathbb{P}^1$ which intersect once for ever supersingular $j$-invariant (see Deligne-Rapoport).
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$\mathcal{W}_A(p^2)$ contains one interesting affinoid $Y_A^1$ which reduces to

$$y^2 = x^{(p+1)/i(A)} - 1$$

(where $i(A) = |\text{Aut}(A)|/2$). $X^+_{11}$ and $X^-_{11}$ reduce to $lg(p)$, while $X_{20}$ and $X_{02}$ are trivial. Picture the stable reduction as ss horizontal components meeting the four vertical (ordinary) ones.
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$\mathcal{W}_A(p^3)$ contains two “old” lifts of $Y_A^1$ (meeting $X_{21}^\pm$ and $X_{12}^\pm$) and one “new” affinoid $Z_A^1$. The latter reduces to a copy of $\mathbb{P}^1$ (called the “bridging component”) crossed by $2(p+1)/i(A)$ copies of $y^2 = x^p - x$. Picture the stable reduction as ss horizontal necklaces, each meeting all six vertical (ordinary) components.
For $p = 23$ there are three supersingular $j$-invariants: $j = 0$, $j = 1728$, and $j = c$ (so $i(A) = 3, 2, \text{ and } 1$ respectively). Therefore, the stable reduction of $X_0(23)$ will consist of two copies of $\mathbb{P}^1$ intersecting in three points (total genus=2).
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$X_0(23^2)$ will have two vertical copies of $lg(23)$ (genus 5), and three horizontal components: $y^2 = x^8 - 1$, $y^2 = x^{12} - 1$, and $y^2 = x^{24} - 1$ (genus 3, 5, and 11). So total genus is 35.
The stable reduction of $X_0(23^3)$ has four vertical copies of $Ig(23)$ and two vertical $\mathbb{P}^1$'s. The reduction of each $W_A(23^3)$ has two lifts of $Y_A^1$ plus a new bridging component, $Z_A^1$, which is crossed by 16, 24, and 48 copies of $y^2 = x^{23} - x$ (respectively).

$$g = (2 \cdot 3 - 1)(3 - 1) + 4(5) + (2 \cdot 3 + 16 \cdot 11)$$
$$+ (2 \cdot 5 + 24 \cdot 11) + (2 \cdot 11 + 48 \cdot 11) = 1036$$
Coleman-McMurdy Strategy for $W_A(p^n)$

**Step 0:** Lift everything from previous level.

**Step 1:** Use moduli-theoretic conditions involving geometry of the $p$-power torsion of $E$ to define an affinoid, $Y_A$ or $Z_A$, in $W_A(p^n)$ which is invariant under the Atkin-Lehner involution $w_n$ (when $A/\mathbb{F}_p$). If $n$ is even this will involve too-supersingular curves. If $n$ is odd, “Atkin-Lehner” or “Self-Dual” curves.

**Step 2:** Use explicit analysis to compute the reduction of $Y_A$ or $Z_A$. In [C-Mc1] we used an approximation formula of de Shalit to do $W_A(p^2)$ and $W_A(p^3)$. We are now using the Gross-Hopkins period map.

**Step 3:** Combine $w_n$ with formal group deformations to construct involutions with $p$ fixed points on the singular residue classes of $Y_A$ or $Z_A$ (leading to $y^2 = x^p - x$ components).

**Step 4:** Compute total genus to show that nothing else interesting can happen.

Today, we focus on $Y_A^2 \subseteq W_A(p^4)$. 
Proposition: Let $S^i$ be the set of all $(x, y) \in X_0(p^i) \times X_0(p^i)$, where $x = (E, C_1)$, $y = (E, C_2)$, and $C_1 \cap C_2 = (0)$. 
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(1) $S^i \cong X_0(p^{2i})$ by $\psi_i : (x, y) \mapsto (E/C_2, p^{-i}C_1/C_2)$. 
Exploitable Symmetry (Even Power Levels)

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2. The following diagram commutes.

\[
\begin{array}{ccc}
S^i & \xrightarrow{\psi_i} & X_0(p^{2i}) \\
\downarrow{\pi_f, \pi_f} & & \downarrow{\pi_{aa}} \\
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3. Atkin-Lehner just switches $x$ and $y$.

4. Let $Y^i_A \subseteq W_A(p^{2i})$ be the subspace whose points correspond to pairs $(E, C)$ where $E/p^iC$ is too-supersingular. Let $TS^i_A \subseteq W_A(p^i)$ be the subspace corresponding to pairs $(E, C)$ where $E$ is too-supersingular. Then $\psi_i$ identifies $Y^i_A$ with $(TS^i_A \times TS^i_A) \cap S_i$ (so in particular, it’s invariant under $w_n$).
We can identify $W_A(p)$ with $\{ t \mid 0 < v_p(t) < 1 \}$ and $W_A(1)$ with $\{ s \mid v(s) > 0 \}$, such that $TS_A^1$ is the circle where $v(t) = \frac{p}{p+1}$ and the forgetful map, $\pi_f : W_A(p) \rightarrow W_A(1)$ is given by

$$s = \pi_f(t) \equiv t + \left( \frac{\kappa}{t} \right)^p$$

for some $\kappa \in W(\mathbb{F}_{p^2})$ with $v(\kappa) = 1$. 
We can identify $W_A(p)$ with $\{ t \mid 0 < \nu_p(t) < 1 \}$ and $W_A(1)$ with $\{ s \mid \nu(s) > 0 \}$, such that $TS_A^1$ is the circle where $\nu(t) = \frac{p}{p+1}$ and the forgetful map, $\pi_f : W_A(p) \to W_A(1)$ is given by

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for some $\kappa \in W(\mathbb{F}_{p^2})$ with $\nu(\kappa) = 1$.

Let $x, y = \alpha^p / t$ for any $\alpha$ with $\nu(\alpha) = \frac{1}{p+1}$. Then $x, y$ identify $TS_A^1$ with the unit circle, and the reduction of $Y_A^1$ is given by

$$x^{-1} - y^{-1} \equiv (x^p - y^p)(\kappa / \alpha^{p+1})^p \pmod{\alpha}.$$ 

Then let $s = 1 / (y - x)$ and $r = (y + x) / (y - x)$ to get

$$s^{p+1} = c(r^2 - 1).$$
Conjecture for $Y^2_A$

The too-supersingular disk of $W_A(1)$ lifts to $p + 1$ residue classes of $TS^i(A)$ for all $i$. So this leads to $p(p + 1)$ residue classes in $Y^i_A$ that will contain all CM points.

We have a way to construct involutions on $p + 1$ of them when $i = 2$ (so inside $W_A(p^4)$). Conjecturally, we expect the reduction of $Y^2_A$ to be birational to the reduction of $Y^1_A$ but with $p + 1$ singular residue classes, each of which contains $p$ copies of $y^2 = x^p - x$. 
$X_0(25)$ and “Symmetric Model” for $X_0(625)$

$Y_A^1$ is the affinoid where $\nu(u) = 1/2$ and $\nu(u^2 - 5) = 1$. Fake CM residue classes are the ten classes where $\nu(u^{12} - 5^6) > 6$. 
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The $\pi_{1,1}$ extension to $X_0(625)$ is given by

$$\frac{x^5}{x^4 + 5x^3 + 15x^2 + 25x + 25} = u(u^4 + 5u^3 + 15u^2 + 25u + 25)$$

$$\frac{y^5}{y^4 + 5y^3 + 15y^2 + 25y + 25} = \frac{5^3}{u^5}(u^4 + 5u^3 + 15u^2 + 25u + 25)$$
Nearly Stable Model for $X_0(125)$

$$f(X, Y) = X^5 - Y^4 - 5XY^3 - 15X^2Y^2 - 25X^3Y - 25X^4 - 5Y^3$$
$$- 5XY^2 + 25X^3 - 15Y^2 - 25X^2 - 25Y + 25X - 25 = 0$$

$$XZ^2 - YZ + 5 = 0$$
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Isomorphism with $(u, y)$ model is given explicitly by

$$\sigma(u, y) = \left(\frac{uy}{5}, u + y, \frac{5}{y}\right) \quad \sigma^{-1}(X, Y, Z) = (XZ, 5/Z).$$
Analysis

\[ f(X, Y) = X^5 - Y^4 - 5XY^3 - 15X^2Y^2 - 25X^3Y - 25X^4 - 5Y^3 \]
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**Proposition:** Let \( A \) be the subaffinoid of \( X_0(125)^+ \) which is determined by \( \nu(Y) = 1/2 \) and \( \nu(Y^2 - 5) = 1 \). Then \( A \) can be parameterized using \( \{ s \in C[0] \mid \nu(s^2 + 1) = 0 \} \) and the map:

\[ X = (\sqrt[5]{-5}s^2 + \sqrt[5]{-5})^2 \quad Y \approx \sqrt{-5}s^5. \]
Analysis

\[ f(X, Y) = X^5 - Y^4 - 5XY^3 - 15X^2Y^2 - 25X^3Y - 25X^4 - 5Y^3 \\
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**Proposition:** Let \( A \) be the subaffinoid of \( X_0(125)^+ \) which is determined by \( v(Y) = 1/2 \) and \( v(Y^2 - 5) = 1 \). Then \( A \) can be parameterized using \( \{ s \in C[0] \mid v(s^2 + 1) = 0 \} \) and the map:

\[ X = (\sqrt[5]{-5}s^2 + \sqrt[5]{-5})^2 \quad Y \approx \sqrt{-5}s^5. \]

**Proof** Use Hensel’s Lemma to construct the map from the (punctured) unit circle into \( A \). i.e. Choose \( \alpha \in \mathbb{C}_5 \) such that \( \alpha^{10} = -5 \). Let \( \hat{f}(X, Y) = \frac{1}{25}f(\alpha^4X, \alpha^5Y) \), and consider \( g(Y) := \hat{f}((s^2 + 1)^2, Y) \) as a polynomial over the coordinate ring of \( C[0] \). We claim that \( g \) has a root close to \( Y = s^5 \).
More Analysis... (up to $\nu = 1.2$)

\[
X = \alpha^4(s^2 + 1)^2
\]

\[
Y = \alpha^5 \left(s^5 - \frac{10}{s^5}(s^8 + 2s^6 + 2s^4 + s^2) - \frac{5(s^4 - 2)^5}{s^5(1 + s^2)^{15}}\right)
- \frac{\alpha^9(1 + s^2)^2}{s^5} + \frac{\alpha^{10}(1 - s^2)^5}{(1 + s^2)^5}
+ \frac{2\alpha^{13}(1 + s^2)^9}{s^{15}} - \frac{\alpha^{14}(s^2 - 2)^5}{(1 + s^2)^3} + \cdots
\]

\[
u = \alpha^5 s^5 + \frac{\alpha^{10}(1 - s^2)^5}{(1 + s^2)^5} + \frac{2\alpha^{13}(1 + s^2)^9}{s^{15}} - \frac{\alpha^{14}(1 + s^2)^7}{s^{10}} + \cdots
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\[
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An Interesting Component

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\frac{x^5}{x^4 + 5x^3 + 15x^2 + 25x + 25} = 25\alpha^5 s^5 (1 + s^2)^{10} + 25\alpha^{10} (1 - s^4)^5
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\[+ \frac{2 \cdot 25\alpha^{13} (1 + s^2)^{14}}{s^{15}} - \frac{25\alpha^{14} (1 + s^2)^{12}}{s^{10}} + \ldots \]
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\]

Letting \(x = \alpha^9 x_1\) we get

\[
x_1^5 \equiv s^5(1 + s^2)^{10} + \alpha^5(1 - s^4)^5 + \frac{2\alpha^8(1 + s^2)^{14}}{s^{15}} + 2\alpha^8 s^5(1 + s^2)^{10} x_1^2 \pmod{\alpha^9}
\]
An Interesting Component

\[
\frac{x^5}{x^4 + 5x^3 + 15x^2 + 25x + 25} = 25\alpha^5 s^5 (1 + s^2)^{10} + 25\alpha^{10} (1 - s^4)^5
\]

\[+ \frac{2 \cdot 25\alpha^{13}(1 + s^2)^{14}}{s^{15}} - \frac{25\alpha^{14}(1 + s^2)^{12}}{s^{10}} + \cdots \]

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\[x_1^5 \equiv s^5 (1 + s^2)^{10} + \alpha^5 (1 - s^4)^5 + \frac{2\alpha^8 (1 + s^2)^{14}}{s^{15}} + 2\alpha^8 s^5 (1 + s^2)^{10} x_1^2 \quad (\text{mod } \alpha^9)\]

Now let \(x_1 = \beta x_2 + s(1 + s^2)^2 + \alpha(1 - s^4)\), where \(\beta^5 = \alpha^8\), to get

\[x_2^5 \equiv \frac{2(1 + s^2)^{14}}{s^{15}} + 2s^7 (1 + s^2)^{14} \quad (\text{mod } \alpha)\]

\[\equiv 2(1 + s^2)^{14} \frac{(1 + s^{22})}{s^{15}} \quad (\text{mod } \alpha)\]

\[x_3^5 \equiv 1 - s^2 + s^4 - + \cdots + s^{20} \quad (\text{mod } \alpha)\]

This curve is birational to \(P^1\) with two singular points at \(s = \pm 1\)!
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1. $Y^2_A$ is indeed birational to $Y^1_A$. 
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1. $Y^2_A$ is indeed birational to $Y^1_A$.
2. We haven’t missed a way to construct copies of $y^2 = x^p - x$ outside of the $(p + 1)/i(A)$ residue classes that we know about.
$X_0(625)$ example suggests:

1. $Y_A^2$ is indeed birational to $Y_A^1$.

2. We haven’t missed a way to construct copies of $y^2 = x^p - x$ outside of the $(p + 1)/i(A)$ residue classes that we know about.

With closer analysis we should find 5 copies of $y^2 = x^5 - x$ in each of the two residue classes where $s = \pm 1$. At that point, we will pull back to the Gross-Hopkins-like model and attempt to translate calculations to the general case.