Stable Models and $U_p$ Slope Calculations

Ken McMurdy
Ramapo College of New Jersey
U.C. Berkeley Number Theory Seminar

November 16, 2011
Overview of Talk

Part I
- Slopes of \( U_7 \) Acting on Modular Forms for \( \Gamma_{1}(49) \)
  (1) Recall Basic Definitions
  (2) State Theorem of Kilford-McMurdy
  (3) Explicit Examples
  (4) Proof Sketch

Part II
- Optimal Models for \( X_{0}(p^n) \) for Slope Calculations
  (1) Wish List
  (2) A Potentially Useful Family
  (3) Some properties and an Example
Overview of Talk

**Part I** - Slopes of $U_7$ Acting on Modular Forms for $\Gamma_1(49)$

1. Recall Basic Definitions
2. State Theorem of Kilford-McMurdy
3. Explicit Examples
4. Proof Sketch
Overview of Talk

Part I - Slopes of $U_7$ Acting on Modular Forms for $\Gamma_1(49)$

(1) Recall Basic Definitions
(2) State Theorem of Kilford-McMurdy
(3) Explicit Examples
(4) Proof Sketch

Part II - Optimal Models for $X_0(p^n)$ for Slope Calculations

(1) Wish List
(2) A Potentially Useful Family
(3) Some properties and an Example
Basic Definitions

\[ M_k(\Gamma_1(N)), S_k(\Gamma_1(N)) : \text{classical modular forms and cuspforms} \]
\[ M_k(\Gamma_1(N), \epsilon), S_k(\Gamma_1(N), \epsilon) : \text{subspaces with specified character} \]
Basic Definitions

\[ M_k(\Gamma_1(N)), S_k(\Gamma_1(N)) \]: classical modular forms and cuspforms

\[ M_k(\Gamma_1(N), \epsilon), S_k(\Gamma_1(N), \epsilon) \]: subspaces with specified character

When a prime \( p \) divides \( N \), recall that the Hecke operator, \( U_p \), acts on \( M_k(\Gamma_1(N)) \), preserving these subspaces. The action of \( U_p \) on q-expansions at infinity is given by

\[
U_p \left( \sum a_n q^n \right) = \sum a_{np} q^n.
\]
Basic Definitions

\( M_k(\Gamma_1(N)), S_k(\Gamma_1(N)) \): classical modular forms and cuspforms
\( M_k(\Gamma_1(N), \epsilon), S_k(\Gamma_1(N), \epsilon) \): subspaces with specified character

When a prime \( p \) divides \( N \), recall that the Hecke operator, \( U_p \), acts on \( M_k(\Gamma_1(N)) \), preserving these subspaces. The action of \( U_p \) on \( q \)-expansions at infinity is given by

\[
U_p \left( \sum a_n q^n \right) = \sum a_{np} q^n.
\]

Now, let \( f \) be a normalized eigenform defined over a number field \( K \), so that \( a_p \) is its \( U_p \) eigenvalue. Embed \( K \) into \( \mathbb{C}_p \). Then the slope of \( f \) is the \( p \)-adic valuation of \( a_p \) where \( v(p) = 1 \).
Basic Definitions

$M_k(\Gamma_1(N)), S_k(\Gamma_1(N))$: classical modular forms and cuspforms

$M_k(\Gamma_1(N), \epsilon), S_k(\Gamma_1(N), \epsilon)$: subspaces with specified character

When a prime $p$ divides $N$, recall that the Hecke operator, $U_p$, acts on $M_k(\Gamma_1(N))$, preserving these subspaces. The action of $U_p$ on $q$-expansions at infinity is given by

$$U_p \left( \sum a_n q^n \right) = \sum a_{np} q^n.$$

Now, let $f$ be a normalized eigenform defined over a number field $K$, so that $a_p$ is its $U_p$ eigenvalue. Embed $K$ into $\mathbb{C}_p$. Then the **slope** of $f$ is the $p$-adic valuation of $a_p$ where $\nu(p) = 1$.

**Note:** The slope depends on both $f$ and the embedding into $\mathbb{C}_p$. 
Basic Definitions

\( M_k(\Gamma_1(N)), S_k(\Gamma_1(N)) \): classical modular forms and cuspforms

\( M_k(\Gamma_1(N), \epsilon), S_k(\Gamma_1(N), \epsilon) \): subspaces with specified character

When a prime \( p \) divides \( N \), recall that the Hecke operator, \( U_p \), acts on \( M_k(\Gamma_1(N)) \), preserving these subspaces. The action of \( U_p \) on \( q \)-expansions at infinity is given by

\[
U_p \left( \sum a_n q^n \right) = \sum a_{np} q^n.
\]

Now, let \( f \) be a normalized eigenform defined over a number field \( K \), so that \( a_p \) is its \( U_p \) eigenvalue. Embed \( K \) into \( \mathbb{C}_p \). Then the slope of \( f \) is the \( p \)-adic valuation of \( a_p \) where \( v(p) = 1 \).

**Note:** The slope depends on both \( f \) and the embedding into \( \mathbb{C}_p \).

**Open Problem:** Determine the slopes of \( M_k(\Gamma_1(N), \epsilon) \), as a function of \((p, k, N, \epsilon)\) and the embedding.
Kilford-McMurdy for $\Gamma_1(49)$

Fix a primitive $42^{\text{nd}}$ root of unity, $\zeta$, and let $\chi$ be the Dirichlet character of conductor 49 defined by $\chi(3) = \zeta$. Let $K_1$ and $K_2$ be the 7-adic completions of $\mathbb{Q}[[\zeta]]$ so that $v(\zeta + 1) > 0$ and $v(\zeta + 4) > 0$ respectively ($e = 6$ for both).
Fix a primitive 42\textsuperscript{nd} root of unity, \( \zeta \), and let \( \chi \) be the Dirichlet character of conductor 49 defined by \( \chi(3) = \zeta \). Let \( K_1 \) and \( K_2 \) be the 7-adic completions of \( \mathbb{Q}[\zeta] \) so that \( \nu(\zeta + 1) > 0 \) and \( \nu(\zeta + 4) > 0 \) respectively (\( e = 6 \) for both).

(1) \( S_k(\Gamma_1(49), \chi^{7k-6}) \) is diagonalized by \( U_7 \) over \( K_1 \). The slopes of \( U_7 \) on this space are the values less than \( k - 1 \) in

\[
\left\{ \frac{1}{6} \cdot \left\lfloor \frac{9i}{7} \right\rfloor : i \in \mathbb{N} \right\} = \left\{ \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{5}{6}, \ldots \right\}.
\]
Kilford-McMurdy for $\Gamma_1(49)$

Fix a primitive $42^{\text{nd}}$ root of unity, $\zeta$, and let $\chi$ be the Dirichlet character of conductor 49 defined by $\chi(3) = \zeta$. Let $K_1$ and $K_2$ be the 7-adic completions of $\mathbb{Q}[\zeta]$ so that $v(\zeta + 1) > 0$ and $v(\zeta + 4) > 0$ respectively ($e = 6$ for both).

(1) $S_k(\Gamma_1(49), \chi^{7k-6})$ is diagonalized by $U_7$ over $K_1$. The slopes of $U_7$ on this space are the values less than $k - 1$ in

$$\left\{ \frac{1}{6} \cdot \left\lfloor \frac{9i}{7} \right\rfloor : i \in \mathbb{N} \right\} = \left\{ \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{5}{6}, \ldots \right\}.$$

(2) $S_k(\Gamma_1(49), \chi^{8-7k})$ is diagonalized by $U_7$ over $K_2$. The slopes of $U_7$ on this space are the values less than $k - 1$ in

$$\left\{ \frac{1}{6} \cdot \left\lfloor \frac{9i+6}{7} \right\rfloor : i \in \mathbb{N} \right\} = \left\{ \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{6}{6}, \ldots \right\}.$$

(Each slope corresponds to a one dimensional eigenspace.)
Example 1

Let $\psi(3) = \gamma$ a primitive 21st root of unity. Then $S_2(\Gamma_1(49), \psi)$ has one family defined over $\mathbb{Q}(\gamma, \alpha)$ where $\alpha$ is a root of

$$p(x) = x^4 + (\gamma^5 + 1)x^3 + (\gamma^{10} - 5\gamma^5 + 1)x^2$$
$$+ (\gamma^{11} - 4\gamma^{10} - \gamma^7 - \gamma^6 - 2\gamma^5 - \gamma^3 + 2\gamma^2 - \gamma)x$$
$$+ (2\gamma^{10} + \gamma^9 + \gamma^8 + \gamma^7 - \gamma^6 - \gamma^5 - \gamma^4 + \gamma^2 + \gamma + 1).$$
Example 1

Let \( \psi(3) = \gamma \) a primitive 21\(^{st} \) root of unity. Then \( S_2(\Gamma_1(49), \psi) \) has one family defined over \( \mathbb{Q}(\gamma, \alpha) \) where \( \alpha \) is a root of

\[
p(x) = x^4 + (\gamma^5 + 1)x^3 + (\gamma^{10} - 5\gamma^5 + 1)x^2 \\
+ (\gamma^{11} - 4\gamma^{10} - \gamma^7 - \gamma^6 - 2\gamma^5 - \gamma^3 + 2\gamma^2 - \gamma)x \\
+ (2\gamma^{10} + \gamma^9 + \gamma^8 + \gamma^7 - \gamma^6 - \gamma^5 - \gamma^4 + \gamma^2 + \gamma + 1).
\]

The value of \( a_7 \) is then given explicitly by

\[
a_7 = (\gamma^{11} - \gamma^{10} + \gamma^8 - \gamma^7 - \gamma^6 + \gamma^5 - \gamma^3 + \gamma^2 - 1)\alpha^3 \\
+ (\gamma^8 - \gamma^6 + \gamma^5 - \gamma^4 - \gamma^3 + \gamma^2)\alpha^2 \\
+ (4\gamma^{11} - \gamma^6 + \gamma^5 + 4\gamma^4 - \gamma^3 + \gamma^2 - \gamma)\alpha \\
-(\gamma^{11} - \gamma^{10} - 3\gamma^9 + \gamma^8 - \gamma^7 - 2\gamma^6 + 2\gamma^5 + \gamma^4 - 3\gamma^3 + 2\gamma^2 + \gamma - 3).
\]
Example 1

Let $\psi(3) = \gamma$ a primitive $21^{\text{st}}$ root of unity. Then $S_2(\Gamma_1(49), \psi)$ has one family defined over $\mathbb{Q}(\gamma, \alpha)$ where $\alpha$ is a root of

$$p(x) = x^4 + (\gamma^5 + 1)x^3 + (\gamma^{10} - 5\gamma^5 + 1)x^2$$
$$+ (\gamma^{11} - 4\gamma^{10} - \gamma^7 - \gamma^6 - 2\gamma^5 - \gamma^3 + 2\gamma^2 - \gamma)x$$
$$+ (2\gamma^{10} + \gamma^9 + \gamma^8 + \gamma^7 - \gamma^6 - \gamma^5 - \gamma^4 + \gamma^2 + \gamma + 1).$$

The value of $a_7$ is then given explicitly by

$$a_7 = (\gamma^{11} - \gamma^{10} + \gamma^8 - \gamma^7 - \gamma^6 + \gamma^5 - \gamma^3 + \gamma^2 - 1)\alpha^3$$
$$+ (\gamma^8 - \gamma^6 + \gamma^5 - \gamma^4 - \gamma^3 + \gamma^2)\alpha^2$$
$$+ (4\gamma^{11} - \gamma^6 + \gamma^5 + 4\gamma^4 - \gamma^3 + \gamma^2 - \gamma)\alpha$$
$$-(\gamma^{11} - \gamma^{10} - 3\gamma^9 + \gamma^8 - \gamma^7 - 2\gamma^6 + 2\gamma^5 + \gamma^4 - 3\gamma^3 + 2\gamma^2 + \gamma - 3).$$

The theorem applies over $K_1$ if we take $\gamma = \zeta^8$, since

$$\chi^{7(2)-6} = \chi^8 = \psi.$$
Choose the uniformizer $\pi_1 = -\zeta^8 + \zeta^6 - \zeta^4 + \zeta$ for $K_1$. Then $\nu(\pi_1) = 1/6$. The roots of $p(x)$ are defined over $K_1$ with the following approximations:

\[
\begin{align*}
\alpha_1 & = 4 + 5\pi_1 + 1\pi_1^2 + 2\pi_1^3 + 3\pi_1^4 + 5\pi_1^5 + \cdots \\
\alpha_2 & = 5 + 4\pi_1 + 2\pi_1^2 + 3\pi_1^3 + 4\pi_1^4 + 1\pi_1^5 + \cdots \\
\alpha_3 & = 4 + 1\pi_1 + 5\pi_1^2 + 4\pi_1^3 + 1\pi_1^4 + 6\pi_1^5 + \cdots \\
\alpha_4 & = 5 + 5\pi_1^2 + 4\pi_1^3 + 4\pi_1^5 + 2\pi_1^6 + \cdots 
\end{align*}
\]
Choose the uniformizer $\pi_1 = -\zeta^8 + \zeta^6 - \zeta^4 + \zeta$ for $K_1$. Then $v(\pi_1) = 1/6$. The roots of $p(x)$ are defined over $K_1$ with the following approximations:

\[
\begin{align*}
\alpha_1 &= 4 + 5\pi_1 + 1\pi_1^2 + 2\pi_1^3 + 3\pi_1^4 + 5\pi_1^5 + \cdots \\
\alpha_2 &= 5 + 4\pi_1 + 2\pi_1^2 + 3\pi_1^3 + 4\pi_1^4 + 1\pi_1^5 + \cdots \\
\alpha_3 &= 4 + 1\pi_1 + 5\pi_1^2 + 4\pi_1^3 + 1\pi_1^4 + 6\pi_1^5 + \cdots \\
\alpha_4 &= 5 + 5\pi_1^2 + 4\pi_1^3 + 4\pi_1^5 + 2\pi_1^6 + \cdots 
\end{align*}
\]

Plugging these values into $a_7$ we find $\pi_1$-adic valuations of 1, 2, 3, and 5. So the theorem is verified in this special case.
Example 2

Let \( \psi(3) = \gamma \), a primitive 7\textsuperscript{th} root. This time there are three eigenforms to consider in \( S_2(\Gamma_1(49), \psi) \). The theorem will apply over \( K_2 \) if we take \( \gamma = \zeta^{-6} \), since

\[
\chi^{8-7(2)} = \chi^{-6} = \psi.
\]

Working in \( K_2 \), we take as our uniformizer

\[
\pi_2 = \zeta^9 + \zeta^8 + \zeta^4 + \zeta^3 - \zeta - 1.
\]

(Again \( v(\pi_2) = 1/6 \).)
Example 2

Let $\psi(3) = \gamma$, a primitive 7th root. This time there are three eigenforms to consider in $S_2(\Gamma_1(49), \psi)$. The theorem will apply over $K_2$ if we take $\gamma = \zeta^{-6}$, since

$$\chi^{8-7(2)} = \chi^{-6} = \psi.$$ 

Working in $K_2$, we take as our uniformizer

$$\pi_2 = \zeta^9 + \zeta^8 + \zeta^4 + \zeta^3 - \zeta - 1.$$ 

(Again $v(\pi_2) = 1/6$.)

The first form is defined over $\mathbb{Q}(\gamma)$ with $a_7 = 2\gamma^5 + 2\gamma^4 + \gamma^3 + 2$, and so it is easy to check that for this form $v_{\pi_2}(a_7) = 3$. 
Example 2 (cont)

The other two eigenforms are defined over $\mathbb{Q}(\gamma, \alpha)$ where $\alpha$ is a root of

$$p(x) = x^2 - (\gamma^4 + \gamma)x - (\gamma^5 - \gamma^2 - \gamma).$$

Then $a_7$ for either of these forms is given explicitly by

$$a_7 = (\gamma^3 - \gamma^2)\alpha - (\gamma^4 - \gamma^2 - \gamma + 1).$$
Example 2 (cont)

The other two eigenforms are defined over $\mathbb{Q}(\gamma, \alpha)$ where $\alpha$ is a root of

$$p(x) = x^2 - (\gamma^4 + \gamma)x - (\gamma^5 - \gamma^2 - \gamma).$$

Then $a_7$ for either of these forms is given explicitly by

$$a_7 = (\gamma^3 - \gamma^2)\alpha - (\gamma^4 - \gamma^2 - \gamma + 1).$$

The roots of $p(x)$ are defined over $K_2$ with the following approximations:

$$\alpha_1 = 1 + \pi_2 + 6\pi_2^2 + 2\pi_2^3 + \pi_2^4 + \cdots$$
$$\alpha_2 = 1 + 3\pi_2 + 6\pi_2^2 + 3\pi_2^3 + \pi_2^4 + \cdots.$$

Setting $\alpha = \alpha_1$, we have $v_{\pi_2}(a_7) = 4$, and for $\alpha = \alpha_2$ we get $v_{\pi_2}(a_7) = 2$. Thus the theorem is again verified.
Outline of the Proof

In order to use $p$-adic analysis to prove results about slopes of classical modular eigenforms for $\Gamma_1(N)$, these are the standard steps:

1. Use the geometry of $X_1(N)$ to embed $M_k(\Gamma_1(N))$ into a natural $p$-adic family of modular forms ("overconvergent" modular forms). Verify that $U_p$ extends to the family.
2. Compute the $U_p$ slopes of all of the overconvergent eigenforms in your family using analytic techniques.
3a. Use a theorem of Coleman to conclude that all of the overconvergent eigenforms of small slope are classical. Thus, you have constructed a certain number of classical eigenforms with specified slopes.
3b. Compare with known formulas for the total number of classical eigenforms with a given character.
4. Keep fingers crossed that (3a) and (3b) are the same!!
Outline of the Proof

In order to use $p$-adic analysis to prove results about slopes of classical modular eigenforms for $\Gamma_1(N)$, these are the standard steps:

(1) Use the geometry of $X_1(N)$ to embed $M_k(\Gamma_1(N))$ into a natural $p$-adic family of modular forms ("overconvergent" modular forms). Verify that $U_p$ extends to the family.
Outline of the Proof

In order to use $p$-adic analysis to prove results about slopes of classical modular eigenforms for $\Gamma_1(N)$, these are the standard steps:

(1) Use the geometry of $X_1(N)$ to embed $M_k(\Gamma_1(N))$ into a natural $p$-adic family of modular forms (“overconvergent” modular forms). Verify that $U_p$ extends to the family.

(2) Compute the $U_p$ slopes of all of the overconvergent eigenforms in your family using analytic techniques.

(3a) Use a theorem of Coleman to conclude that all of the overconvergent eigenforms of small slope are classical. Thus, you have constructed a certain number of classical eigenforms with specified slopes.

(3b) Compare with known formulas for the total number of classical eigenforms with a given character.

(4) Keep fingers crossed that (3a) and (3b) are the same!!
Outline of the Proof

In order to use $p$-adic analysis to prove results about slopes of classical modular eigenforms for $\Gamma_1(N)$, these are the standard steps:

(1) Use the geometry of $X_1(N)$ to embed $M_k(\Gamma_1(N))$ into a natural $p$-adic family of modular forms ("overconvergent" modular forms). Verify that $U_p$ extends to the family.

(2) Compute the $U_p$ slopes of all of the overconvergent eigenforms in your family using analytic techniques.

(3a) Use a theorem of Coleman to conclude that all of the overconvergent eigenforms of small slope are classical. Thus, you have constructed a certain number of classical eigenforms with specified slopes.

(3b) Compare with known formulas for the total number of classical eigenforms with a given character.

(4) Keep fingers crossed that (3a) and (3b) are the same!!
Outline of the Proof

In order to use $p$-adic analysis to prove results about slopes of classical modular eigenforms for $\Gamma_1(N)$, these are the standard steps:

(1) Use the geometry of $X_1(N)$ to embed $M_k(\Gamma_1(N))$ into a natural $p$-adic family of modular forms (“overconvergent” modular forms). Verify that $U_p$ extends to the family.

(2) Compute the $U_p$ slopes of all of the overconvergent eigenforms in your family using analytic techniques.

(3a) Use a theorem of Coleman to conclude that all of the overconvergent eigenforms of small slope are classical. Thus, you have constructed a certain number of classical eigenforms with specified slopes.

(3b) Compare with known formulas for the total number of classical eigenforms with a given character.
Outline of the Proof

In order to use $p$-adic analysis to prove results about slopes of classical modular eigenforms for $\Gamma_1(N)$, these are the standard steps:

(1) Use the geometry of $X_1(N)$ to embed $M_k(\Gamma_1(N))$ into a natural $p$-adic family of modular forms ("overconvergent" modular forms). Verify that $U_p$ extends to the family.

(2) Compute the $U_p$ slopes of all of the overconvergent eigenforms in your family using analytic techniques.

(3a) Use a theorem of Coleman to conclude that all of the overconvergent eigenforms of small slope are classical. Thus, you have constructed a certain number of classical eigenforms with specified slopes.

(3b) Compare with known formulas for the total number of classical eigenforms with a given character.

(4) Keep fingers crossed that (3a) and (3b) are the same!!
Overconvergent Modular Forms of Level 49

Our overconvergent forms are defined *intrinsically*, following Coleman’s construction in “Classical and Overconvergent Modular Forms of Higher Level.”
Our overconvergent forms are defined *intrinsically*, following Coleman's construction in "Classical and Overconvergent Modular Forms of Higher Level."

Let \( f : E_1(49) \to X_1(49) \) be the universal generalized elliptic curve over \( X_1(49) \), and \( \omega = f_*\Omega^1_{E_1(49)/X_1(49)} \). Then \( M_k(\Gamma_1(49)) \) is just the holomorphic sections of \( \omega \otimes^k \).
Our overconvergent forms are defined *intrinsically*, following Coleman’s construction in “Classical and Overconvergent Modular Forms of Higher Level.”

Let $f : E_1(49) \to X_1(49)$ be the universal generalized elliptic curve over $X_1(49)$, and $\omega = f_* \Omega^1_{E_1(49)/X_1(49)}$. Then $M_k(\Gamma_1(49))$ is just the holomorphic sections of $\omega \otimes^k$.

Let $W = W_1(49)$ be the rigid subspace of $X_1(49)$ corresponding to pairs $(E, P)$ where $< P >$ is the canonical subgroup of order 49. This is a wide open neighborhood of the cusp $\infty$. Following Coleman, we define our overconvergent forms to be holomorphic sections of the analytification of $\omega \otimes^k$ over $W$, denoted by $M_k(\Gamma_1(49))(W)$. 

Overconvergent Modular Forms of Level 49

Our overconvergent forms are defined *intrinsically*, following Coleman’s construction in “Classical and Overconvergent Modular Forms of Higher Level.”

Let $f : E_1(49) \to X_1(49)$ be the universal generalized elliptic curve over $X_1(49)$, and $\omega = f_* \Omega^1_{E_1(49)/X_1(49)}$. Then $M_k(\Gamma_1(49))$ is just the holomorphic sections of $\omega \otimes^k$.

Let $W = W_1(49)$ be the rigid subspace of $X_1(49)$ corresponding to pairs $(E, P)$ where $< P >$ is the canonical subgroup of order 49. This is a wide open neighborhood of the cusp $\infty$. Following Coleman, we define our overconvergent forms to be holomorphic sections of the analytification of $\omega \otimes^k$ over $W$, denoted by $M_k(\Gamma_1(49))(W)$.

Then $U_7$ acts on this space and preserves character subspaces which we denote by $M_k(\Gamma_1(49), \psi)(W)$. 
Translating to $X_0(49)$

We want to work on $X_0(49)$, because we have good explicit equations. This can be done by using Eisenstein series as in Coleman’s “p-adic Banach Spaces” [§B3].
Translating to $X_0(49)$

We want to work on $X_0(49)$, because we have good explicit equations. This can be done by using Eisenstein series as in Coleman's "p-adic Banach Spaces" [§B3].

Let $D$ be the wide open disk in $X_0(49)$ over which $W$ lies via the forgetful map. Let $M_0(\Gamma_0(49))(D)$ denote the holomorphic functions on $D$. Let $\chi$ and $\tau$ be characters of conductor 49 and 7, defined by $\chi(3) = \zeta$ and $\tau(3) = \beta$ (a primitive 6th root). Then we can define an isomorphism

$$M_0(\Gamma_0(49))(D) \cong M_k(\Gamma_1(49), \chi \tau^{k-1})(W),$$

by $f \mapsto f \cdot E_{1,\chi} \cdot E_{1,\tau}^{k-1}$. 
Translating to $X_0(49)$

We want to work on $X_0(49)$, because we have good explicit equations. This can be done by using Eisenstein series as in Coleman’s “p-adic Banach Spaces” [§B3].

Let $D$ be the wide open disk in $X_0(49)$ over which $W$ lies via the forgetful map. Let $M_0(\Gamma_0(49))(D)$ denote the holomorphic functions on $D$. Let $\chi$ and $\tau$ be characters of conductor 49 and 7, defined by $\chi(3) = \zeta$ and $\tau(3) = \beta$ (a primitive 6th root). Then we can define an isomorphism

$$M_0(\Gamma_0(49))(D) \cong M_k(\Gamma_1(49), \chi \tau^{k-1})(W),$$

by $f \mapsto f \cdot E_{1,\chi} \cdot E_{1,\tau}^{k-1}$.

**Note:** Strictly speaking, we would need a holomorphicity factor too, but the character/embedding pairs are chosen so that both Eisenstein series are holomorphic and non-vanishing over $W$. 
Let \( \bar{U}_7 \) be the induced linear operator on \( M_0(\Gamma_0(49))(D) \).
The Explicit Part of the Proof

Now we consider the following explicit model for $X_0(49)$.

\[
y^2 - 7xy(x^2 + 5x + 7) - x(x^6 + 7x^5 + 21x^4 + 49x^3 + 147x^2 + 343x + 343) = 0
\]
\[
z^2 = x(4x^2 + 21x + 28)
\]
\[
x = \eta_1/\eta_{49} \quad y = \eta_7^4/\eta_{49}^4
\]
The Explicit Part of the Proof

Now we consider the following explicit model for $X_0(49)$.

$$y^2 - 7xy(x^2 + 5x + 7) - x(x^6 + 7x^5 + 21x^4 + 49x^3 + 147x^2 + 343x + 343) = 0$$

$$z^2 = x(4x^2 + 21x + 28)$$

$$x = \eta_1/\eta_{49} \quad y = \eta_7^4/\eta_{49}^4$$

Let $\alpha = \sqrt[4]{-7}$. Setting $z = 2\alpha^3 Z$ and $x = \alpha^2 X$, we obtain a good reduction model $\mathcal{X}$ for $X_0(49)$ which reduces to

$$Z^2 = X(X^2 - 1).$$
The Explicit Part of the Proof

Now we consider the following explicit model for $X_0(49)$.

\[
y^2 - 7xy(x^2 + 5x + 7) - x(x^6 + 7x^5 + 21x^4 + 49x^3 + 147x^2 + 343x + 343) = 0
\]

\[
z^2 = x(4x^2 + 21x + 28)
\]

\[
x = \eta_1/\eta_{49} \quad y = \eta_7^4/\eta_{49}^4
\]

Let $\alpha = \sqrt[4]{-7}$. Setting $z = 2\alpha^3 Z$ and $x = \alpha^2 X$, we obtain a good reduction model $\mathcal{X}$ for $X_0(49)$ which reduces to

\[
Z^2 = X(X^2 - 1).
\]

$D$ is just the infinite residue disk on this model. $t = x^4/y$ is a parameter on $X_0(7)$ with divisor $(0) - (\infty)$ which lifts to a parameter on $D$. The parameter $s = \alpha/t$ identifies $D$ with the unit disk. So $M_0(\Gamma_0(49))(D)$ has “basis” $\{s, s^2, s^3, \cdots\}$. 
A Truncation of the Large Matrix ($k = 1$ shown)

Write $\tilde{U}_7(s^i)$ as a power series in $s$, and put the coefficients in the $i$th column. This yields an infinite dimensional matrix that represents $\tilde{U}_7$ in the basis $\{s, s^2, \ldots\}$. A truncation of the corresponding matrix of 7-adic valuations, over $K_1$, is as follows.

$$
\begin{bmatrix}
1/6 & 5/12 & 1/2 & 3/4 & 1 & 5/4 & 3/2 \\
1/4 & 1/3 & 7/12 & 5/6 & 13/12 & 7/6 & 29/12 \\
1/6 & 5/12 & 2/3 & 11/12 & 1 & 5/4 & 5/2 \\
1/4 & 1/2 & 3/4 & 5/6 & 13/12 & 4/3 & 31/12 \\
1/6 & 5/12 & 1/2 & 3/4 & 1 & 5/4 & 7/3 \\
1/4 & 1/3 & 7/12 & 5/6 & 13/12 & 7/6 & 29/12 \\
1/6 & 5/12 & 2/3 & 11/12 & 1 & 5/4 & 5/2
\end{bmatrix}
$$

Our theorem says that the sequence of slopes should be $\{1/6, 1/3, 1/2, 5/6, 1\}$ (almost the sequence of column valuations). This will follow if the determinant of each $j \times j$ truncation is larger than that of any other principle $j \times j$ minor. To prove that, we consider the “column functions.”
A Truncation of the Large Matrix ($k = 1$ shown)

Write $\tilde{U}_7(s^i)$ as a power series in $s$, and put the coefficients in the $i^{th}$ column. This yields an infinite dimensional matrix that represents $\tilde{U}_7$ in the basis $\{s, s^2, \ldots\}$. A truncation of the corresponding matrix of 7-adic valuations, over $K_1$, is as follows.

$$
\begin{bmatrix}
1/6 & 5/12 & 1/2 & 3/4 & 1 & 5/4 & 3/2 \\
1/4 & 1/3 & 7/12 & 5/6 & 13/12 & 7/6 & 29/12 \\
1/6 & 5/12 & 2/3 & 11/12 & 1 & 5/4 & 5/2 \\
1/4 & 1/2 & 3/4 & 5/6 & 13/12 & 4/3 & 31/12 \\
1/6 & 5/12 & 1/2 & 3/4 & 1 & 5/4 & 7/3 \\
1/4 & 1/3 & 7/12 & 5/6 & 13/12 & 7/6 & 29/12 \\
1/6 & 5/12 & 2/3 & 11/12 & 1 & 5/4 & 5/2
\end{bmatrix}
$$

Our theorem says that the sequence of slopes should be

$\{\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{5}{6}, 1, \frac{7}{6}, \frac{3}{2}, \ldots\}$ (almost the sequence of column valuations). This will follow if the determinant of each $j \times j$ truncation is larger than that of any other principle $j \times j$ minor. To prove that, we consider the “column functions.”
Proposition: Approximations for $\tilde{U}_7(s^i)$ for $1 \leq i \leq 7$ over $K_1(\alpha)$ where $\alpha^4 = -7$ are as follows.

\[
\begin{align*}
\tilde{U}_7(s^1) &\equiv 2\alpha\pi_1 z/(x(x + \pi_1^3)), & v_1 = 2, & e_1 \geq 3 \\
\tilde{U}_7(s^2) &\equiv 4\alpha^2\pi_1^2 / x, & v_1 = 4, & e_1 \geq 5 \\
\tilde{U}_7(s^3) &\equiv \alpha^3 z/x^2 + 5\alpha^3\pi_1^2 / x, & v_1 = 6, & e_1 \geq 8 \\
\tilde{U}_7(s^4) &\equiv 3\alpha^4 z/x^2 + 2\alpha^4\pi_1^2 (x + 4\pi_1^3) / x^2, & v_1 = 9, & e_1 \geq 11 \\
\tilde{U}_7(s^5) &\equiv 6\alpha^5 z(x + \pi_1^3) / x^3, & v_1 = 12, & e_1 \geq 13 \\
\tilde{U}_7(s^6) &\equiv \alpha^6\pi_1 (x^2 + 7) / x^3, & v_1 = 14, & e_1 \geq 15 \\
\tilde{U}_7(s^7) &\equiv \alpha^7 / t, & v_1 = 18, & e_1 \geq 19
\end{align*}
\]

(A recursive formula kicks in from there.)

Note: $\frac{1}{12} v_1(f)$ denotes the minimal 7-adic valuation of $f$ over $D$. 

Scaling and reducing the column functions on the stable reduction, we have the following functions and divisors.

\[
\frac{Z}{X(X-1)} = (\infty) + (-1,0) - (0,0) - (1,0)
\]
\[
\frac{1}{X} = 2(\infty) - 2(0,0)
\]
\[
\frac{Z}{X^2} = (1,0) + (-1,0) + (\infty) - 3(0,0)
\]
\[
\frac{(X-1)}{X^2} = 2(1,0) + 2(\infty) - 4(0,0)
\]
\[
\frac{Z(X-1)}{X^3} = 3(1,0) + (-1,0) + (\infty) - 5(0,0)
\]
\[
\frac{(X^2-1)}{X^3} = 2(1,0) + 2(-1,0) + 2(\infty) - 6(0,0)
\]
\[
\frac{Z(X^2-1)}{X^4} = 3(1,0) + 3(-1,0) + (\infty) - 7(0,0).
\]

By Riemann-Roch, no linear combination of the first \(j\) can ever vanish to degree \(j + 1\) at \(\infty\). Thus, the determinant of the \(j^{th}\) truncation approximates the \(j^{th}\) coefficient of the characteristic series and the slopes are as claimed.
Part II - Optimal Models for $X_0(p^n)$ for Slope Calculations
Optimal Models for $X_0(p^n)$ for Slope Calculations

In order to make a similar slope argument more generally, we would need a model with the following properties.
Optimal Models for $X_0(p^n)$ for Slope Calculations

In order to make a similar slope argument more generally, we would need a model with the following properties.

(1) We must be able to write down a “Banach basis” for the functions on $W_1(p^n)$.

Canonical Example: Let $W$ be the wide open in $P_1$ whose $\mathbb{C}$-valued points satisfy $v((x−1)(x−2)(x−3))<1$ (the complement of three affinoid disks). Then $A_K(W) = \mathbb{K} < X, Y, Z > / (XY − p(X−Y), 2XZ − p(X−Z), YZ − p(Y−Z))$.

Think $X = p^t−1$, $Y = p^t−2$, and $Z = p^t−3$ for a parameter $t$ on $P_1$. In general, $W_1(p^n)$ is isomorphic to the complement in $P_1$ of ss affinoid disks (one for each supersingular $j$-invariant).
In order to make a similar slope argument more generally, we would need a model with the following properties.

(1) We must be able to write down a “Banach basis” for the functions on $W_1(p^n)$.

**Canonical Example:** Let $W$ be the wide open in $\mathbb{P}^1$ whose $\mathbb{C}_p$-valued points satisfy

$$v((x - 1)(x - 2)(x - 3)) < 1$$

(the complement of three affinoid disks). Then

$$A_K(W) = K < X, Y, Z > / (XY - p(X - Y), 2XZ - p(X - Z), YZ - p(Y - Z)).$$

Think $X = \frac{p}{t-1}$, $Y = \frac{p}{t-2}$ and $Z = \frac{p}{t-3}$ for a parameter $t$ on $\mathbb{P}^1$.

In general, $W_1(p^n)$ is isomorphic to the complement in $\mathbb{P}^1$ of ss affinoid disks (one for each supersingular $j$-invariant).
(2) Parameters should generate the Weierstrass parameters on the “first” supersingular components.

Stable reduction of $X_0(p^3)$ when $p = 12k + 11$ is shown. The left-most genus 0 vertical component is the reduction of $W_1(p^3)$. It intersects the components, $Y^A_{2,1}$, which have the equation

$$y^2 = x^{(p+1)/i(A)} - 1.$$
Candidate Model for $X_0(p)$

$$
t = \left( \frac{\eta_1}{\eta_p} \right)^{e_1} \quad x = \left( \frac{dt/t}{(\eta_1 \eta_p)^2} \right)^{e_2}
$$

If $p = 12k + 1$, we have: $(e_1, e_2) = (2, 6)$ and

$$(t) = k(0) - k(\infty)$$

$(x)_{neg} = -(6k + 1)(0) - (6k + 1)(\infty)$. 

If $p = 12k + 5$, we have $(e_1, e_2) = (6, 2)$ and

$$(t) = (3k + 1)(0) - (3k + 1)(\infty)$$

$(x)_{neg} = -(2k + 1)(0) - (2k + 1)(\infty)$. 

If $p = 12k + 7$, we have $(e_1, e_2) = (4, 3)$ and

$$(t) = (2k + 1)(0) - (2k + 1)(\infty)$$

$(x)_{neg} = -(3k + 2)(0) - (3k + 2)(\infty)$. 

If $p = 12k + 11$, we have $(e_1, e_2) = (12, 1)$ and

$$(t) = (6k + 5)(0) - (6k + 5)(\infty)$$

$(x)_{neg} = -(k + 1)(0) - (k + 1)(\infty)$. 

Properties and Example

**Important Fact:** The Atkin-Lehner involution, $w_1$, fixes $x$ and satisfies

$$w_1^* t \equiv \frac{p(e_1/2)}{t}.$$

**Example:** $X_0(17)$ has the equation:

$$t^3 x^4 + (-3934t^3)x^3 + (-8608t^4 + 2667641t^3 - 42291104t^2)x^2$$
$$+ (-2944t^5 - 408968t^4 - 38771644t^3 - 2009259784t^2 - 71061003136t)x$$
$$- 256t^6 - 79328t^5 - 11950529t^4 - 1059834654t^3$$
$$- 58712948977t^2 - 1914785073632t - 30358496383232 = 0$$

It's actually much nicer. For example, $f(0, t) = t^3 \cdot [g(t) + g(\frac{17^3}{t})]$, where

$$g(t) = -256t^3 - 79328t^2 - 11950529t - 529917327.$$