Appetizers

1. A bracelet consists of 6 identical spherical beads. Any two consecutive beads on the bracelet are tangent to one another. When the bracelet is placed on a flat surface, the distance from the center of the bracelet (the center of the beads arrangement) to the center of any of the beads is $c$. Find the volume of the bracelet in terms of $c$. Simplify your answer.

2. Find $\int_0^1 \sqrt{x(1-x)} \, dx$.

3. How many squares are there with sides parallel to x and y axes and integer vertices $(i, j)$, where $0 \leq i \leq n$ and $0 \leq j \leq n$?

4. Find positive integers $x$ and $y$ such that $x^2 + y^2 = 2018$.

Entrees

5. The numbers $1!, 2!, 3!, \ldots, 60!$ are written on a blackboard. One of these 60 numbers is erased and it turns out that the product of the remaining numbers on the blackboard is a perfect square. Which number was erased? (You may write your answer in the form $n!$. You do not have to evaluate the factorial.)

6. Sharks have long lifespans. An 80-year-old shark has 45% chance of living at least 20 more years and 9% chance of living at least 30 more years. A 90-year-old shark has a 12% chance of living at least 20 more years. Find the probability that a 90-year-old shark lives to see her 100-th birthday.

7. Find a positive integer $n$ divisible by 2 and by 9 and having exactly 10 factors including 1 and $n$.

8. Find the sum $\sum_{k=1}^{\infty} \frac{k+4}{k(k+1)(k+2)}$.

Desserts

9. Consider a triangle $ABC$ with median $AD$. Let $G$ be a point on the median $AD$ such that $\frac{AG}{GD} = \frac{5}{2}$ and let $E$ be the point of intersection of line $BG$ with the side $AC$. Find the proportion $\frac{\text{Area } \triangle ABE}{\text{Area } \triangle BEC}$. 
10. The white king is positioned at a1 on a chessboard. In how many different ways can it move to h8 passing through d3 and e6 if each single step can be either up or to the right?

11. The polynomial $p(x)$ has degree 5, the leading coefficient 1 and satisfies $p(1) = 1$, $p(2) = 3$, $p(3) = 6$, $p(4) = 10$, and $p(5) = 15$. Find $p(6)$.

12. Find the function $y = f(x)$ on the interval $(1, \infty)$ that is the solution of the initial value problem \( \frac{dy}{dx} = \frac{1}{\sqrt{x} - \sqrt{x}} \) subject to the initial condition $f(64) = 41$. 
Appetizers

1. Cylindrical cans, each containing 3 tennis balls are stacked in a pyramid with triangular base. Each level of the pyramid consists of cans arranged tightly in an equilateral triangle. There are $n$ cans along each side of the bottom level triangle. The side of each higher level triangle is by one can shorter than the prior one. Finally, there is just one can on the top level. If each tennis ball has radius 1, what is the total volume of all tennis balls? Express your answer as a closed formula in terms of $n$.

Hint: $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$.

2. A machine dispenses M&Ms of 4 different colors: Blue, Green, Red and Yellow. M&Ms of different colors are loaded into the machine in a mixed bulk and they are dispensed at random into small bags of 12 each.
   a. How many different bag contents are possible, if a bag is to contain at least one M&M of each color?
   b. How many different bag contents are possible, if a bag does not have to include at least one M&M of each color?

Entrees

3. Find all possible triangles such that one of the sides has length 5, the other two sides have integer lengths and one of the angles is $60^\circ$.

4. A Moon expedition project assumes building a base camp on a flat plot of the Moon surface. The base will consist of two semispherical structures connected with a tunnel in the shape of a half cylinder. The distance between the centers of hemispheres will be 30 meters. The radius of each hemisphere will be 5 meters and the radius of the semi-cylindrical tunnel will be 3 meters. Find the volume of the air needed to fill the base.

Desserts

5. Adam, Billy, Chris and Danny are playing the following ball game. Adam begins by throwing a ball at Billy. If Billy can catch it then next it is his turn to throw the ball at Chris. However, if Billy does not catch it, he is out of the game and it is then Chris’s turn to throw the ball at Danny. Again, if Danny fails to catch the ball he will be out of the game and it will be Adams turn to throw the ball at the next person still in the game. If Danny catches the ball, it would be his turn to throw the ball at Adam. The game continues until one person is left in the game. Assume that all players have the same probability $\frac{1}{2}$ of catching the ball and their successes or failures are independent. Find the probability that Adam is the winner.

6. Find the dimensions of a rectangular prism box with a square base and the smallest volume that will hold 107 balls each having 1-inch diameter.