Appetizers:

1. How many 1cm × 1cm × 1cm cubes are needed to build the shell portion of a cube for which the area of each face is 16 cm²? (The resulting solid will be hollow inside.)

2. Find the value of $x + y$, if
   \[
   \frac{x - 2016}{y - 2015} + \frac{y - 2015}{x - 2016} = -2. 
   \]

3. How many integers, $0 \leq x < 2^8$, have the property that there are no consecutive 1’s in their binary expansion? For example, 74 has this property while 181 does not.

   \[
   74 = (1001010)_2 \quad 181 = (10110101)_2 
   \]

4. Let $y = f(x)$ be the unique solution on the interval $(0, 9)$ to the following initial value problem.

   \[
   \frac{dy}{dx} = \sqrt{\frac{9 - x}{x}} \quad y(5) = 0 
   \]

   Compute the arc length along this curve from $x = 1$ to $x = 4$.

Entrees:

5. Two real numbers $a$ and $b$ are selected independently at random from the interval $[0, 1]$. I.e., the values of $a$ and $b$ are uniformly distributed over the interval. What is the probability that the quadratic equation $x^2 + ax + b = 0$ has real roots?

6. A right triangle is drawn so that its vertex is at the origin, its legs lie along the non-negative coordinate axes, and its hypotenuse is tangent to the graph of $y = -\ln x$. (See figure below.) Determine the maximum area of this triangle.
7. An independent farmer specializes in green beans, which he grows in two fields. One field is extremely reliable, yielding a predictable 20 bushels for every bag of seed. The other field is more risky, yielding 60 bushels per bag with probability 4/5 but only 10 bushels per bag with probability 1/5. The farmer has 20 bags of seed and does not want to risk dropping below a total yield of \( Y = 320 \) bushels (for both fields). So he decides to plant in such a way as to maximize the expected value of \( \ln(Y - 320) \). How many bags of seed will he plant in the risky field?

8. Determine the smallest positive integer \( x \) for which \( x^3 + 9x + 81 \) is divisible by 243.

Desserts:

9. A math talk on Diophantine equations was given in a large lecture hall which housed a rectangular array of chairs. After everyone in attendance took a seat, the speaker (who is not seated) noticed there were exactly 7 men seated in each row, exactly 9 women seated in each column, and exactly 2 empty chairs. At least how many people attended the talk (not including the speaker)?

10. Evaluate \( \lim_{n \to \infty} \sum_{k=1}^{99} \left( \sin^n \frac{k! \pi}{1001} + \cos^n \frac{k! \pi}{1001} \right) \).

11. The formula for all nonzero entries of an \( n \times n \) matrix \( A_n \) is given below, along with \( A_5 \) for reference. Compute \( \lim_{n \to \infty} \det A_n \) (the limit of the determinant of \( A_n \)).

\[
a_{i,j} = \begin{cases} 
1, & \text{if } i = j = 1 \\
\frac{1}{n}, & \text{if } i = j > 1 \\
\frac{n+1}{n}, & \text{if } i = j + 1 \\
(-1)^{j+1} n, & \text{if } i = 1, j > 1 
\end{cases}
\]

\[
A_5 = \begin{bmatrix}
1 & -5/6 & 5/6 & -5/6 & 5/6 \\
6/5 & 1/5 & 0 & 0 & 0 \\
0 & 6/5 & 1/5 & 0 & 0 \\
0 & 0 & 6/5 & 1/5 & 0 \\
0 & 0 & 0 & 6/5 & 1/5 
\end{bmatrix}
\]

12. Ben and Jerry are shooting baskets. Every time Ben takes a shot, he has a \( \frac{1}{3} \) probability of making it. Every time Jerry takes a shot, his chance of making it is \( \frac{2}{3} \). They decide to have a competition where Ben takes exactly 5 shots and Jerry takes exactly \( m \) shots. What is the minimum value of \( m \) for which the probability that Ben makes at least 4 more baskets than Jerry is greater than or equal to \( \frac{1}{3^{m+1}} \)?
Appetizers:

1. Find the sum \( \sum_{k=1}^{2016} (-1)^{1+2+\cdots+k} k. \)

2. Find the real number \( b, 0 \leq b \leq 1, \) for which the area between the curves \( y = x^2 \) and \( y = b \) from \( x = 0 \) to \( x = 1 \) is a minimum. What is this minimum area? (It is the shaded area in the figure below.)

\[ \int_0^1 (\sin^{-1} x)^2 \, dx. \]

Entrees:

3. Define a sequence \( (b_n) \) by \( b_1 = 1 \) and \( b_{n+1} = b_n + \frac{1}{b_n} \) for all \( n \geq 1. \) Does \( (b_n) \) converge? Justify your answer.

4. Suppose that for some \( t > 0, f : [0, t] \to \mathbb{R} \) is a continuous, strictly increasing function that passes through the origin. Let \( g : [0, f(t)] \to \mathbb{R} \) be the inverse function, which must exist.

   a. Prove that
   \[
   \int_0^{f(t)} g(x)^2 \, dx = 2 \int_0^t x(f(t) - f(x)) \, dx.
   \]

   b. Use Part (a) to compute
   \[
   \int_0^1 (\sin^{-1} x)^2 \, dx.
   \]

Desserts:

5. Marty and Emmett have two identical but independent time machines. For either machine, regardless of the programmed arrival time, \( t_0, \) the actual arrival time \( t \) (in hours) follows the probability density function \( f(t) = e^{-2|t-t_0|}. \) They decide to travel back in time separately to witness the exact moment on November 5th, 1955, when time travel was first discovered. What is the probability that Marty arrives early, Emmett arrives late, but they arrive within one hour of each other?
6. A container in the shape of an inverted cone with radius 1 ft and height 1 ft has negligible weight, but is initially filled to the top with sand weighing 16 pounds. The container is attached to several helium balloons, which provide a continuous upward force of 15 pounds, and released from the top of a very tall building. Suppose there is a leak at the bottom of the container, causing the depth of the sand at the center of the container, \( h(t) \), to decrease at an initial rate of \( \frac{1}{64} \text{ ft/sec} \). After 7 seconds, the container has not hit the ground. What is its velocity at that time?

Assume that the height \( y(t) \) of the container above the ground is governed by Newton’s Second Law of Motion, \( m y'' = 15 - mg \), where \( g = 32 \text{ ft/sec}^2 \) is the acceleration of gravity and \( m = m(t) \) is the mass of the sand at time \( t \). Assume that \( h(t) \) follows the modified Torricelli’s Law:

\[
A \frac{dh}{dt} = -k \sqrt{2(g + y'')}h,
\]

where \( A = A(t) \) is the cross-sectional area at the top of the sand at time \( t \) and \( k \) is some constant.