Appetizers:

1. The sum of the positive divisors of 36, including 1 and itself, is 91. What is the sum of the reciprocals of those positive divisors?

**Solution:**

\[ \sum_{d \mid 36} \frac{1}{d} = \frac{1}{36} \sum_{d \mid 36} 36 = \frac{1}{36} \sum_{d \mid 36} d = 91 \]

2. Compute \( \int_{0}^{4} (x - 2) + (x - 2)^2 + (x - 2)^3 + (x - 2)^4 + (x - 2)^5 \, dx \).

**Solution:** Using symmetry, we can eliminate three terms and compute the integral as follows.

\[ 2 \int_{2}^{4} (x - 2)^2 + (x - 2)^4 \, dx = 2 \left[ \frac{1}{3} (x - 2)^3 + \frac{1}{5} (x - 2)^5 \right]_{2}^{4} = 2 \left( \frac{8}{3} + \frac{32}{5} \right) = \frac{272}{15} \]

3. The decimal representation of \( \frac{1}{17} \) is 0.0588235294117647. Determine the decimal representation of \( \frac{1}{588235294117647} \).

**Solution:** The given information may be rewritten in the following form.

\[ 588235294117647(10^{-16} + 10^{-32} + \cdots) = \frac{1}{17} \]

By simply switching the 17 and the 588235294117647, we see that \( 1/588235294117647 = 0.00000000000000017 \).

4. Let \( P = (-2, 2), \ Q = (1, -3), \) and \( R = (5, 4) \). There are exactly three points in the plane, call them \( S_1, S_2 \) and \( S_3 \), such that \( \{P, Q, R, S_i\} \) is the set of vertices of a parallelogram. Find the value of

\[ \frac{\text{Area}(\triangle S_1 S_2 S_3)}{\text{Area}(\triangle PQR)} \]

**Solution:** The points, \( P, Q \) and \( R \) are essentially irrelevant. Each of the three parallelograms is formed by gluing a rotated copy of triangle \( \triangle PQR \) to itself, along one of the three edges. Hence, the area of \( \triangle S_1 S_2 S_3 \) is precisely 4 times the area of \( \triangle PQR \). (See figure below.)

![Diagram](image)

5. Determine the unique value of \( x \) for which

\[ \lim_{n \to \infty} (1 + x)(1 + x^2)(1 + x^4)(1 + x^8) \cdots (1 + x^{2^n}) = 2014. \]

**Solution:** This is simply a different way to write the finite geometric series, \( 1 + x + x^2 + x^3 + \cdots + x^{2^n+1} - 1 \). If \( |x| > 1 \) or \( x = 1 \), the limit clearly does not exist. The limit is 0 if \( x = -1 \). In all other cases, the limit exists and is given by

\[ \lim_{n \to \infty} \frac{1 - x^{2^n+1}}{1 - x} = \frac{1}{1 - x} \]

Setting this equal to 2014, we find the unique solution to be \( x = \frac{2013}{2014} \).
Entrees:

6. Find the shortest possible length of an interval \([a, b]\) for which \(\int_a^b \frac{4}{1 + x^2} \, dx = \pi\). You may not leave any transcendental functions in your answer.

Solution: Computing the integral, we find that \(\tan^{-1} b = \tan^{-1} a + \frac{\pi}{4}\). So then we may take the tangent of both sides to obtain

\[ b = \tan \left( \tan^{-1} a + \frac{\pi}{4} \right) = \frac{a + 1}{1 - a}. \]

Therefore, as a function of \(a\), the required length of the interval \([a, b]\) would be

\[ \ell(a) = b - a = \frac{a + 1}{1 - a} - a = 1 + \frac{a^2}{1 - a}. \]

Setting \(\ell'(a) = 0\) to find the minimum, we find that \(a^2 - 2a - 1 = 0\). This yields \(a = 1 \pm \sqrt{2}\), but only \(a = 1 - \sqrt{2}\) results in \(b > a\). So the minimum length is given by \(\ell(1 - \sqrt{2}) = 2\sqrt{2} - 2\).

On the other hand, symmetry of the graph suggests that \(a = -b\). So we need only solve

\[ \int_0^b \frac{4}{1 + x^2} \, dx = \frac{\pi}{2}. \]

This results in an interval of length \(2b = 2\tan(\pi/8)\), which is easily shown to equal \(2\sqrt{2} - 2\) with the standard formula for \(\tan(2\theta)\).

7. Compute \(\lim_{n \to \infty} \left( \sqrt[3]{8n^3 + 4n^2 + n + 11} - \sqrt[6]{4n^2 + n + 9} \right)\).

Solution: Standard arguments show that

\[ \lim_{n \to \infty} \left( \sqrt[3]{8n^3 + 4n^2 + n + 11} - \sqrt[6]{4n^2 + n + 9} \right) = 1/3 \quad \text{and} \quad \lim_{n \to \infty} \left( \sqrt[6]{4n^2 + n + 9} - 2n \right) = 1/4. \]

So the limit in question must exist and equal \(\frac{1}{3} - \frac{1}{4} = \frac{1}{12}\).

8. A dart, thrown at random, hits a 1 ft \(\times\) 1 ft square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the dart hits a point that is within 1 ft of every corner of the dartboard.

Solution: Identifying the dartboard with the unit square, \([0, 1] \times [0, 1]\), the region in question is bounded by four circles: \(x^2 + y^2 = 1\), \((x - 1)^2 + y^2 = 1\), \(x^2 + (y - 1)^2 = 1\) and \((x - 1)^2 + (y - 1)^2 = 1\). (See figure below).

![Figure](image.png)

By symmetry we can compute the area of the region, which equals the probability since the total area of the square is 1, to be:

\[
4 \int_{1/2}^{\sqrt{3}/2} \sqrt{1 - x^2} - \frac{1}{2} \, dx = 4 \int_{1/2}^{\sqrt{3}/2} \sqrt{1 - x^2} \, dx - 4 \int_{1/2}^{\sqrt{3}/2} \frac{1}{2} \, dx \\
= 4 \int_{\pi/6}^{\pi/3} \cos^2 \theta \, d\theta - \sqrt{3} + 1 \quad (x = \sin \theta) \\
= 4 \left[ \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_{\pi/6}^{\pi/3} - \sqrt{3} + 1 = \frac{\pi}{3} - \sqrt{3} + 1.
\]
9. It is straightforward to show that the following function is continuous at \( x = 0 \). Is it differentiable there? If so, find the derivative.

\[
 f(x) = \begin{cases} 
 (e^x \sin x - x^2) / x^2, & \text{if } x \neq 0 \\
 1, & \text{if } x = 0 
\end{cases}
\]

**Solution:** One way to do this is straight from the definition of the derivative.

\[
 f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{(e^x \sin x - x^2) / x^2 - 1}{x} = \lim_{x \to 0} \frac{e^x \sin x - x^2}{x^3} = \frac{1}{3} 
\]

The last limit can be computed either with l'Hospital's Rule or by using the Maclaurin series for \( e^x \) and \( \sin x \). On the other hand, plugging in the Maclaurin series from the start, we see that \( f(x) \) is an analytic function given on \((-\infty, \infty)\) by

\[
 f(x) = 1 + \frac{1}{3} x - \frac{1}{30} x^3 - \frac{1}{60} x^4 + \cdots 
\]

Differentiating this power series gives the same answer for \( f'(0) \).

10. A non-empty set of positive integers is said to be square-valued if the product of all of its elements is a perfect square. How many (non-empty) subsets of \([15, 21, 35, 42, 66, 110]\) are square-valued?

**Solution:** First factor the integers.

\[
 15 = 3 \cdot 5, \quad 21 = 3 \cdot 7, \quad 35 = 5 \cdot 7, \quad 42 = 2 \cdot 3 \cdot 7, \quad 66 = 2 \cdot 3 \cdot 11, \quad 110 = 2 \cdot 5 \cdot 11 
\]

Any product of some subset of these numbers will be a square if and only if the total exponent for each of the five support primes is even. This reasoning leads to the following three solutions.

\[
 \{21, 35, 66, 110\}, \quad \{15, 21, 35\}, \quad \{15, 66, 110\} 
\]

A more sophisticated approach would be to view the exponents of the primes (for each number) as vectors in \((\mathbb{F}_2)^5\). When we multiply the numbers, we add the corresponding vectors, and we get a square precisely if the sum of the vectors is 0. Hence the problem is equivalent to computing the nullspace of the following matrix.

\[
 \begin{bmatrix}
 0 & 0 & 0 & 1 & 1 & 1 \\
 1 & 1 & 0 & 1 & 1 & 0 \\
 1 & 0 & 1 & 0 & 0 & 1 \\
 0 & 1 & 1 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 \\
 0 & 0 & 0 & 0 & 1 & 1 \\
 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}
 1 & 0 & 1 & 0 & 0 & 1 \\
 0 & 1 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 \\
 \end{bmatrix}
\]

Thus, we have a two dimensional nullspace spanned by \( \vec{a} = [1 1 1 0 0 0]^T \) and \( \vec{b} = [1 0 0 0 1]^T \). These correspond to the first and third subsets listed above, and the middle subset corresponds to \( \vec{a} + \vec{b} \).

**Desserts:**

11. The base of a pyramid is a right triangle whose smallest angle is \( \alpha \). The right triangle is inscribed in a circle, and then a cone is formed using the circle as its base and the top of the pyramid as its vertex. Find, in terms of \( \alpha \), the ratio of the volume of the pyramid to the volume of the cone.

**Solution:** Because the pyramid and cone have the same height, their volumes will have the same ratio as the areas of their bases. Note also that the hypotenuse must be a diameter of the circle by basic geometry. So, in terms of the radius \( r \), the two legs of the right triangle will necessarily be \( 2r \sin \alpha \) and \( 2r \cos \alpha \). This means that the ratio of the areas is given by

\[
 \frac{(1/2)(2r \sin \alpha)(2r \cos \alpha)}{\pi r^2} = \sin 2\alpha
\]

12. The seat on a child’s swing set hangs at a length of 4 meters from the overhead cross bar (see figure below). For safety, the swing has built-in damping that prevents motion above the horizontal. Pumping his legs as hard as possible, the child finds that his speed, as a function of \( \theta \) (radians), is modeled by the function: \( s(\theta) = 12\sqrt{\cos \theta} \) m/s. With this speed function, at what value of \( \cos \theta \) should the child jump off the swing, if he wants to achieve the maximum total height? Assume, for simplicity, a gravitational acceleration of 10 m/s\(^2\).
Solution: From the diagram, the child leaves the swing with a direction of \( \langle \cos \theta, \sin \theta \rangle \) and a height of \(-4 \cos \theta\) relative to the top of the swing. Call that moment \( t = 0 \). Then from that moment on, the height and vertical velocity are given by

\[
h(t) = -4 \cos \theta + (12 \sqrt{\cos \theta \sin \theta}) t - 5t^2 \quad h'(t) = 12 \sqrt{\cos \theta \sin \theta} - 10t.
\]

Setting \( h'(t) = 0 \), we find that \( t_{\text{max}} = \frac{6 \sqrt{\cos \theta \sin \theta}}{5} \). Then we plug this into \( h(t) \) to find the maximum height that the child achieves, as a function of \( \theta \).

\[
h_{\text{max}}(\theta) = -4 \cos \theta + \frac{72}{5} \cos \theta \sin^2 \theta - \frac{36}{5} \cos \theta \sin^2 \theta = \frac{18}{5} \cos \theta - \frac{36}{5} \cos^3 \theta
\]

Now we simply set \( h'_{\text{max}}(\theta) = 0 \), and solve for the interesting solution, \( \cos(\theta) = \frac{2}{3\sqrt{3}} \).

13. In Super Mega Lotto, six different numbers are selected from 1, 2, \ldots, 99. Which number is most likely to be the second smallest of those selected?

Solution: The second smallest number must satisfy \( 2 \leq k \leq 95 \). For any such \( k \), the probability that it will be the second largest is given by

\[
\left( \frac{k-1}{99} \right) \binom{99}{4} = \frac{(k-1)(99-k)(98-k)(97-k)(96-k)}{4! \cdot \binom{99}{6}}.
\]

Let \( P_k \) be the numerator. Then maximizing the probability is equivalent to maximizing \( P_k \). So now consider the difference between \( P_k \) and \( P_{k+1} \).

\[
P_{k+1} - P_k = k(98-k)(97-k)(96-k)(95-k) - (k-1)(99-k)(98-k)(97-k)(96-k) = (98-k)(97-k)(96-k)(95k-k^2+k^2-100k+99)
\]

Thus, the difference is positive for \( k \leq 19 \) and negative for \( k \geq 20 \), and the maximum probability must occur when \( k = 20 \).

14. A rectangular box is to have a volume of 3 cubic meters, and a diagonal length of \( \sqrt{7} \) meters. Find the maximum and minimum perimeter that such a box can have. Note: Here, the perimeter refers to the sum of the lengths of all twelve edges.

Solution: If we let \( x, y \) and \( z \) represent the dimensions, we can view this as an optimization problem for the function, \( P(x, y, z) = 4x + 4y + 4z \), over the two constraints, \( xyz = 3 \) and \( x^2 + y^2 + z^2 = 7 \). Then the Lagrange Multiplier method produces the following system of equations.

\[
\begin{align*}
4 &= \lambda yz + 2\mu x \\
4 &= \lambda xz + 2\mu y \\
4 &= \lambda xy + 2\mu z \\
3 &= xyz \\
7 &= x^2 + y^2 + z^2
\end{align*}
\]

Using the first constraint, and the fact that \( x, y, z > 0 \), the first three equations may be equivalently rewritten.

\[
\begin{align*}
4x &= 3\lambda + 2\mu x^2 \\
4y &= 3\lambda + 2\mu y^2 \\
4z &= 3\lambda + 2\mu z^2
\end{align*}
\]
Now, if \( \mu = 0 \), this implies that \( x = y = z \), which is impossible since there is no such point on both constraints. Therefore, from the first two we derive that \( x = y = 2/\mu \). We have an analogous statement for \( y \) and \( z \), as well as one for \( x \) and \( z \). The \( \mu \) condition can not be true all three times, as again it would imply \( x = y = z \). Thus, we conclude that exactly two of the variables are equal.

So now we simply plug \( y = x \) into the two constraints and solve simultaneously. This results in two points (each of which represents two others by symmetry).

\[
f(\sqrt{3}, \sqrt{3}, 1) = 8\sqrt{3} + 4 \quad f(\sqrt{6}/2, \sqrt{6}/2, 2) = 4\sqrt{6} + 8
\]

The former is the maximum and the latter is the minimum.

15. Let \( a_n \) denote the integer that is closest to \( \sqrt{n} \), which is well defined, since the fractional part of \( \sqrt{n} \) cannot equal 1/2. Determine the value of the following series.

\[
\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \frac{1}{2^{a_3}} + \frac{1}{2^{a_4}} + \cdots
\]

**Solution:** The condition, \( k - 1/2 < \sqrt{n} < k + 1/2 \), is equivalent to:

\[
k^2 - k + 1/4 < n < k^2 + k + 1/4.
\]

So the number of \( n \) values for which \( a_n = k \) is \( 2k \). (Indeed, we must have \( n = k^2 - k + 1, \ldots, k^2 + k \).) So, instead of summing over \( n \), the series can be rewritten as the following sum over \( k \).

\[
\sum_{k=1}^{\infty} \frac{2k}{2^k} = \frac{2}{2^1} + \frac{4}{2^2} + \frac{6}{2^3} + \cdots
\]

This sum can be computed in a variety of ways. For example, one can differentiate the Maclaurin series for \( 1/(1 - x) \) and substitute \( x = 1/2 \), or obtain a closed form expression for the recursively defined sequence representing the sum.

\[
s_1 = 1, s_{n+1} = s_n + (n + 1)(1/2)^n \quad \rightarrow \quad s_n = 4 - (n + 2)(1/2)^{n-1}
\]

By either method, we see that the sum is 4.
Appetizers:

1. Professor Ubongo comes to class with 24 donuts for his 24 students, who are already seated around a circular table when he arrives. Professor Ubongo chooses a number $k$ from 1 to 23 (inclusive). He then distributes the donuts by going clockwise around the circle, giving one donut to every $k$th student, beginning with his favorite. This procedure works out fairly, as in the end every student has exactly one donut. How many possible values are there for $k$?

Solution: Label the students as 0, 1, 2, ..., 23, starting with Professor Ubongo’s favorite. Then the $n$th donut will be given to the student whose number is $k(n-1) \mod 24$. This function of $n$ will take on all possible values in $\mathbb{Z}_{24}$ if and only if $\gcd(k, 24) = 1$. So there are $\phi(24) = 8$ possible values for $k$, namely 1, 5, 7, 11, 13, 17, 19, and 23.

2. Determine the area of the smallest right triangle with vertex at the origin, whose two legs lie on the non-negative $x$ and $y$ axes, and whose hypotenuse is tangent to the graph of $y = 1 - x^2$ at some point in the first quadrant. (See figure below.)

Solution: Let $(t, 1 - t^2)$ be the point of tangency. From the derivative, the slope at this point is $m = -2t$. Hence, if $b$ and $h$ are the lengths of the base and height, we must have:

$$\frac{h-1+t^2}{b-t} = 2t \quad \rightarrow \quad h = 1 + t^2$$

$$\frac{1-t^2}{b-t} = 2t \quad \rightarrow \quad b = \frac{t}{2} + \frac{1}{2t}.$$  

So the area function of $t$ is given by $A(t) = \frac{1}{4}(t^3 + 2t + 1/t)$. Now set the derivative to 0 to find the minimum.

$$A'(t) = \frac{1}{4}(t^2 + 1)(3t^2 - 1)/t^2 = 0$$

The minimum must occur when $t = 1/\sqrt{3}$, resulting in an area of $4\sqrt{3}/9$.

Entrees:

3. An ancient game begins when 1 cup of rice is distributed over four bowls, situated at the four numbered vertices of a diamond (as pictured below). Once the rice is distributed, each round of the game consists of the following three steps, executed in order.

   Step 1: The rice at $V_1$ is divided evenly and given to $V_2$ and $V_3$ (exactly half to each).
   Step 2: $V_4$ gives all of its rice to $V_1$.
   Step 3: $V_2$ and $V_3$ each give $2/3$ of their rice to $V_4$ (including what they received in Step 1).

The player wins after the completion of the $n$th full round, if the new distribution of rice ever matches the original distribution.

(a) Determine all possible ways to win this game in the very first round.
(b) Show that, in fact, it is only possible to win this game if one wins in the very first round.

Solution: Let $\vec{x} = [x_1 \ x_2 \ x_3 \ x_4]^T$ represent the amount of rice at each vertex. Then each round of the game transforms $\vec{x}$ by the linear function $T(\vec{x}) = A\vec{x}$, where $A$ is the following matrix.

$$
\begin{bmatrix}
0 & 0 & 0 & 1 \\
1/6 & 1/3 & 0 & 0 \\
1/6 & 0 & 1/3 & 0 \\
2/3 & 2/3 & 2/3 & 0
\end{bmatrix}
$$
The vector, \( \bar{x} \), is fixed if and only if it is an eigenvector of \( A \) with eigenvalue 1. Solving for the nullspace of \( A - 1I \), we find a one-dimensional subspace spanned by \([4 \ 1 \ 1 \ 4]^T\). Since \( x_1 + x_2 + x_3 + x_4 \) must equal 1, this leads to the unique solution of \( x_1 = x_4 = \frac{2}{5}, \ x_2 = x_3 = \frac{1}{5} \).

For Part (b), it suffices to compute the remaining eigenvalues of \( A \).

\[
\det(A - \lambda I) = \frac{1}{5} (9\lambda^4 - 6\lambda^3 - 5\lambda^2 + 2\lambda) = \frac{1}{5} \lambda(\lambda - 1)(3\lambda - 1)(3\lambda + 2) = 0
\]

\[\lambda = 0, \ 1, \ \frac{1}{3}, \ -\frac{2}{3}\]

We see that, after \( n \) rounds, the original vector has been transformed by \( T^n \), which has 4 distinct eigenvalues: \( \lambda = 0, \ \lambda = 1, \ \lambda = (1/3)^n \) and \( \lambda = (-2/3)^n \). Hence, the dimension of the \( \lambda = 1 \) eigenspace will always be exactly 1, and there will never be any new solutions beyond the one given in Part (a).

4. Let \( f(x) = \sqrt[3]{x^3 + 8} \). The graphs of \( y = f(x) \) and \( y = f^{-1}(x) \) are given below, along with the graph of \( y = x \) for reference. Is the area between the graphs of \( y = f(x) \) and \( y = f^{-1}(x) \) finite or infinite? You must justify your answer.

**Solution:** First we compute the inverse function to be \( f^{-1}(x) = \sqrt[3]{x^3 - 8} \). Now, by symmetry and continuity, the area between the two graphs will be finite if and only if the following integral converges.

\[
\int_{2}^{\infty} \left[ \sqrt[3]{x^3 + 8} - \sqrt[3]{x^3 - 8} \right] \, dx
\]

Using the difference of cubes formula, \( a^3 - b^3 = (a - b)(a^2 + ab + b^2) \), we see that the following inequality holds on \([2, \infty)\).

\[
0 \leq \sqrt[3]{x^3 + 8} - \sqrt[3]{x^3 - 8} = \frac{(x^3 + 8) - (x^3 - 8)}{(\sqrt[3]{x^3 + 8})^2 + \sqrt[3]{x^3 + 8} \sqrt[3]{x^3 - 8} + (\sqrt[3]{x^3 - 8})^2} \leq \frac{16}{x^2}
\]

Therefore, since the integral of \( 16/x^2 \) on \([2, \infty)\) converges, so does the integral in question, by the simple Comparison Test.

**Desserts:**

5. Zach buys a family mobile plan for himself, his wife Yolanda and his daughter Xena. The plan provides up to 100 free minutes per month for each individual, as long as the total for the whole family does not exceed 220 minutes. Suppose that Zach’s monthly usage is a random variable \( Z \) that is uniformly distributed over the interval \([0, 50]\). Similarly, Xena uses \( X \) minutes per month, and Yolanda uses \( Y \) minutes per month, where \( X \) and \( Y \) are uniformly distributed over \([60, 110]\). Assuming that \( X \), \( Y \) and \( Z \) are independent, compute the probability that Zach’s family will be charged extra for exceeding the allotment of free minutes.

**Solution:** The sample space here is the rectangular solid \( R \) described by \( 60 \leq X \leq 110, \ 60 \leq Y \leq 110 \) and \( 0 \leq Z \leq 50 \). The distribution is uniform over this sample space, so that the probability of an event is simply the volume of that event divided by the total volume of \( R \). We are looking for the probability of the event \( E \) given by \((X > 100) \lor (Y > 100) \lor (Z > 100) \lor (X + Y + Z > 220) \). However, it is much easier to compute the probability of the complementary event:

\[
E^c : (X \leq 100) \land (Y \leq 100) \land (Z \leq 100) \land (X + Y + Z \leq 220)
\]

Note that \( Z \leq 100 \) is satisfied everywhere on \( R \). So computing the probability of this event amounts to figuring out how much volume the last condition actually takes away from the solid: \([60, 100] \times [60, 100] \times [0, 50]\). Thinking geometrically, this condition removes a tetrahedron with the following vertices:

\((100, 100, 50), (70, 100, 50), (100, 70, 50), (100, 100, 20)\).
The volume of this tetrahedron (by Calculus or Geometry) is \( \frac{1}{6}(30)^3 \). So the probability of \( E^c \) is given by

\[
P(E^c) = \frac{40 \cdot 40 \cdot 50 - (1/6)(30)^3}{50 \cdot 50 \cdot 50} = \frac{151}{250}.
\]

Therefore the probability of \( E \) is \( \frac{99}{250} \).

6. Recall that a point \((x,y,z)\) in \( \mathbb{R}^3 \) is a **lattice point** if all three coordinates are integers. Including vertices and points that lie on the edges, how many lattice points lie on the surface of the tetrahedron whose vertices are at \((0,0,0)\), \((50,0,0)\), \((0,20,0)\), and \((0,0,24)\)?

**Solution:** We will count the lattice points on each vertex, edge and face, and then use Inclusion/Exclusion to account for overlap. Of the six edges, the three that coincide with the coordinate axes clearly have 51, 21 and 25 lattice points. The remaining three can be counted with elementary number theory.

\[
\begin{align*}
\# \{(x,y,0) : 2x + 5y = 100\} &= 11 & (x = 0, 5, \ldots, 50) \\
\# \{(x,0,z) : 12x + 25z = 600\} &= 3 & (x = 0, 25, 50) \\
\# \{(0,y,z) : 6y + 5z = 120\} &= 5 & (y = 0, 5, \ldots, 20)
\end{align*}
\]

Now we consider the faces. Three of them are right triangles with legs along the coordinate axes and lattice point vertices. So the lattice points on each of these can be counted easily, since we may flip the triangle over its hypotenuse to form a rectangle.

\[
\begin{align*}
2F_{xy} - 11 &= 51 \cdot 21 & \rightarrow & F_{xy} = 541 \\
2F_{xz} - 3 &= 25 \cdot 51 & \rightarrow & F_{xz} = 639 \\
2F_{yz} - 5 &= 21 \cdot 25 & \rightarrow & F_{yz} = 265
\end{align*}
\]

For the lattice points on the fourth face, we are solving the following equation over the non-negative integers.

\[
12x + 30y + 25z = 600
\]

It is immediately clear that \( z = 0, 6, 12, 18, 24 \), which leads to \( 2x + 5y = n \) where \( n = 100, 75, \ldots, 0 \). Thus, we find a total number of \( 11 + 8 + 6 + 3 + 1 = 29 \) lattice points on this face.

Finally, observe that each edge is adjacent to exactly two faces, while each vertex is adjacent to three edges and three faces. So we correctly count the total number of lattice points on the surface of the tetrahedron to be:

\[
541 + 639 + 265 + 29 - (51 + 21 + 25 + 11 + 3 + 5) + 4 = 1362.
\]