Appetizers:

1. Alice lives $\frac{1}{3}$ of a kilometer away from Bob’s house. If she plans to walk at the speed of 4 kilometers per hour, how many minutes will it take her to walk to Bob’s house? Simplify your answer.

2. Five students took a Calculus exam for which there were 100 possible points. Their scores were 70, 86, 92, 80 and $x$. Helen observed that the median and mean were equal for the five scores. Find all possible values of $x$.

3. A rectangle is inscribed in a regular hexagon, as shown in the figure below. Find the ratio of the area of the rectangle to the area of the hexagon.

4. Arnold the plumber has 10 pockets and 60 nuts. He wants to put the nuts in all of his pockets so that no two pockets have the same number of nuts. What is the largest number of nuts that any one pocket can possess? **Note:** It is possible to leave a pocket empty.

5. Find all pairs of real numbers $(a, b)$ for which the function

$$f(x) = \begin{cases} \frac{ax}{x^3 - 4x^2 + 5x} & \text{if } x \leq b \\ \frac{x^3}{x^3 - 4x^2 + 5x} & \text{if } x > b \end{cases}$$

is differentiable everywhere.

Entrees:

6. Let $a_1, a_2, a_3, \ldots$ be the increasing sequence of positive integers that are not divisible by 2 or 3. The sequence begins 1, 5, 7, 11, 13, 17, …. Compute the sum of the following series.

$$\sum_{n=1}^{\infty} \frac{1}{2^{a_n}} = \frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \frac{1}{2^{a_3}} + \ldots$$

7. Big Ed’s Hardware sells four distinct items: hammer, screwdriver, pliers, and flashlight. Each item has a constant price. For three particular days, Big Ed’s sales sheet shows the following items sold and resulting total revenue (in dollars).

<table>
<thead>
<tr>
<th>Hammers</th>
<th>Screwdrivers</th>
<th>Pliers</th>
<th>Flashlights</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>1</td>
<td>9</td>
<td>39</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0</td>
<td>5</td>
<td>19</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>2</td>
<td>3</td>
<td>45</td>
</tr>
</tbody>
</table>

On the day before a big storm, Big Ed sells 59 flashlights, in addition to 2 screwdrivers, 1 set of pliers, and 12 hammers. How much total revenue did he make on that day, or is it impossible to tell?

8. Let $\{x\}$ denote the fractional part of $x$, that is, $\{x\} = x - n$, where $n$ is the greatest integer that is less than or equal to $x$. For example, $\{2.718\} = .718$, $\{5/2\} = 1/2$, and $\{\pi\} = \pi - 3$. Define the sequence, $a_1, a_2, a_3, \ldots$ by $a_1 = \sqrt{3}$ and $a_{n+1} = \left\{1/a_n\right\}$ for $n = 1, 2, 3, \ldots$. Find the value of $a_{2013}$.
9. Compute the area of the region that is enclosed by the graph of \( r = \cos \left( \frac{3\theta}{2} \right) \) in polar coordinates.

10. For the function, \( f(x) = \ln(1 - 1/x^2) \), find the value of:

\[
f'(2) + f'(3) + f'(4) + \cdots + f'(100).
\]

Desserts:

11. Compute \( \lim_{n \to \infty} \frac{1}{(\ln n)^2} \left( \frac{\ln 3}{3} + \frac{\ln 4}{4} + \cdots + \frac{\ln n}{n} \right) \).

12. Suppose that \( f \) and \( g \) are infinitely differentiable functions on the set of real numbers, satisfying \( f(0) = g(0) = f'(0) = g'(0) = 1 \) and

\[
\begin{align*}
    f''(x)g(x) &= 4f(x)g(x) - f'(x)g'(x) \\
    g''(x)f(x) &= 5f(x)g(x) - f'(x)g'(x).
\end{align*}
\]

If \( f(1) = 1 \), what is the value of \( g(1) \)?

13. Two large ponds on Farmer Smith’s property nearly meet, at a point where their boundaries are closely modeled in an appropriate coordinate system by the parabolas \( x = y^2 \) and \( x = -\left( y - \frac{33}{2} \right)^2 \). Farmer Smith wants to make a straight path, of constant width, that runs between the two ponds (as pictured below). What slope should she choose for the path, in the given coordinate system, if it is to be as wide as possible? **Note:** Your answer will be a rational number.

14. A goblet has the shape of the graph of \( y = x^4 \) revolved around the \( y \)-axis. A spherical olive is dropped into the goblet and touches the very bottom. What is the largest possible radius for the olive?

15. An urn contains \( n \) red balls and one green ball. The following procedure is carried out \( n \) times. First, a ball is selected at random from the urn, and its color is noted. Then, that ball is returned to the urn, along with one additional green ball. Let \( E_n \) denote the expected number of green balls among the \( n \) selected balls. Calculate \( \lim_{n \to \infty} \frac{E_n}{n} \).
Appetizers:

1. A function \( f : \mathbb{N} \rightarrow \mathbb{N} \) is uniquely determined by:
\[
\begin{align*}
   f(1) &= 1 \\
   f(2n) &= 3f(n) \\
   f(2n+1) &= 3f(n) + 1, \quad \forall n \geq 1.
\end{align*}
\]
Determine \( f(2013) \).

2. Over the interval \([1, 2]\), the functions \( f(x) = x^2 + bx + c \) and \( g(x) = x + \frac{2}{x} \) take on the same minimum value. Moreover, this minimum occurs for the same value of \( x \). What is the maximum value of \( f(x) \) over the same interval?

Entrees:

3. Let \( S \) be the set of vertices of a regular 36-gon. What is the smallest value of \( n \) for which any subset of \( S \) of size \( n \) must contain the three vertices of some equilateral triangle? You must prove your answer.

4. For each positive integer \( k \), let
\[
A_k = \begin{bmatrix}
1 & k & 1 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}.
\]
(a) Find a closed form expression for the matrix \( A_k^n \) in terms of \( n \). You must prove your answer.
(b) Find all ordered pairs \((k, n)\) of positive integers for which \( A_k^n = A_{75} \).

Desserts:

5. In preparation for a regional rocket competition, a team of engineering students designs and builds a rocket propulsion system consisting of two elements:

   (1) A slingshot mechanism that gives the rocket an initial upward velocity of 40 ft/sec every time.
   (2) A rocket engine that produces a sustained, constant upward acceleration of \( X \) ft/sec\(^2\) (not counting gravity). Unfortunately, \( X \) is found to vary uniformly from test to test over the interval \([20, 25]\).

To qualify for the competition, the rocket must remain in the air for a minimum of 10 seconds, after being shot upward from a platform that is 100 ft above the ground. What is the probability that the team will qualify for the competition?

Note: Assume that wind resistance is negligible. Take the acceleration of gravity to be \(-32 \text{ ft/sec}^2\).

6. An ant crawling on the Cartesian plane can travel along the \( x \)-axis at a speed of 2 inches per second, but only at a speed of 1 inch per second anywhere else on the plane. Find the area of the set of points in the plane that the ant can reach in 1 second, if it starts at the origin.