1. Find an equation of the tangent line to the curve \( x^2y^2 = (y + 1)^2(4 - y^2) \) at the point \((2\sqrt{3}, 1)\).

   **Solution.** Use implicit differentiation:
   \[
   2xy^2 + x^2 \cdot 2yy' = y' \left[ 2(y + 1)(4 - y^2) + (y + 1)^2 \cdot (-2y) \right].
   \]
   Plug in \((x, y) = (2\sqrt{3}, 1)\), we can easily solve \(y' = -\frac{\sqrt{3}}{2}\). So the desired tangent line is
   \[
   y - 1 = -\frac{\sqrt{3}}{2} (x - 2\sqrt{3}).
   \]

2. Find the 2007th derivative of \(\sin(-\pi x)\).

   **Solution.**
   \[
   (-\pi)^{2007} \cdot (-\cos(-\pi x)) = (-\pi)^{2007} \cos(\pi x).
   \]

3. Solve \(\log_x 4 = \log_2 x = 1, \ 0 < x < 1\).

   **Solution.** Let \(y = \log_2 x\). Then equation becomes \(\frac{2}{y} - y = 1\) by Change of Base formula. So \(y = 1, -2\). Since \(x = 2^y\) and \(0 < x < 1, x = 1/4\).

4. A mathematics competition took place in an auditorium. There are totally 21 rows of seats and each row has 2 more seats than its previous row. Student A, sitting in the 12th row, found out that there were 30 seats in that row. What is the total number of seats in the class room?

   **Solution.** Let \(a_k\) be the number of seats in the \(k^{th}\) row. Note that the seating arrangements form an arithmetic series: \(a_k = a_1 + 2(k - 1)\) for \(1 \leq k \leq 21\). Since the middle row (11th row) would have \(30 - 2 = 28\) seats. Total number of seats in the room:
   \[
   \sum_{k=1}^{21} a_k = \frac{a_1 + a_{21}}{2} \cdot 21 = 28 \times 21 = 588.
   \]
5. Prove that the product of \( n \) consecutive natural numbers is always divisible by \( n! \).

**Solution.** Let \( p(n, k) \) denote the statement that \( n!|(k(k + 1) \cdots (k + n - 1)) \).

(a) It is obvious that \( p(1, k) \) is true for all \( k \in \mathbb{N} \), and that \( p(n, 1) \) is true for all \( n \in \mathbb{N} \).

(b) Assume that \( p(n - 1, k + 1) \) and \( p(n, k) \) are both true, i.e.

\[
(k + 1)(k + 2) \cdots (k + n - 1) = M \cdot (n - 1)!, \quad k(k + 1) \cdots (k + n - 1) = N \cdot n!.
\]

Consider \( p(n, k + 1) \):

\[
(k + 1)(k + 2) \cdots (k + n) = k(k + 1)(k + 2) \cdots (k + n - 1) + n(k + 1)(k + 2) \cdots (k + n - 1)
\]
\[
= N \cdot n! + n \cdot M \cdot (n - 1)!
\]
\[
= (N + M)n!.
\]

So, \( (k + 1)(k + 2) \cdots (k + n) \) is also divisible by \( n \). \( p(n, k + 1) \) is also true.

(c) By mathematical induction, \( p(n, k) \) is true for all \( n, k \in \mathbb{N} \).

6. Find the interval of decrease of \( f(x) = x^{1/x} \) and show that \( 999^{1000} > 1000^{999} \).

**Solution.**

\[
\ln f(x) = \ln \frac{x}{x}
\]
\[
f'(x) = \frac{1 - \ln x}{x^2}
\]
\[
f(x) = x^{\ln x}
\]
\[
f'(x) = \frac{1 - \ln x}{x^2} \cdot x^{1/x}. 
\]

So \( f'(x) \leq 0 \) when \( x > e \). Therefore

\[
999^{1/999} > 1000^{1/1000}
\]
\[
999^{1000} > 1000^{999}.
\]
7. Let \( x \) be the solution of \( x \cdot 3^x = 3^{18} \). Find an integer \( k \) such that \( k \leq x < k + 1 \).

\textbf{Solution.} Notice that \( f(x) = x \cdot 3^x - 3^{18} \) is an increasing function. Verify that \( f(15) < 0 \) and \( f(16) > 0 \), so the root of \( f(x) \) lies between 15 and 16, by Intermediate Value Theorem. So, \( k = 15 \).

8. Evaluate

\[
\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots
\]

\textbf{Solution.} This series is absolutely convergent, by integral test and comparison with \( \int_{1}^{\infty} \frac{1}{t^3} \, dt \).

\[
= \frac{1}{2} \left( \frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} \right) + \frac{1}{2} \left( \frac{1}{2 \cdot 3} - \frac{1}{3 \cdot 4} \right) + \frac{1}{2} \left( \frac{1}{3 \cdot 4} - \frac{1}{4 \cdot 5} \right) + \cdots
\]

\[
= \frac{1}{2} \left( \frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3} - \frac{1}{3 \cdot 4} + \frac{1}{3 \cdot 4} - \frac{1}{4 \cdot 5} + \cdots \right)
\]

\[
= \frac{1}{2} \cdot \frac{1}{1 \cdot 2}
\]

\[
= \frac{1}{4}.
\]

9. Evaluate

\[
\sum_{n=0}^{2007} (n+1)2^n = 1 + 2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \cdots + 2008 \cdot 2^{2007}.
\]

\textbf{Solution.}

\[
1 + x + \cdots + x^{2008} = \frac{x^{2009} - 1}{x - 1}
\]

\[
\frac{d}{dx} \left( 1 + x + \cdots + x^{2008} \right) = \frac{d}{dx} \left( \frac{x^{2009} - 1}{x - 1} \right)
\]

\[
1 + 2x + 3x^2 + \cdots + 2008x^{2007} = \frac{2009x^{2008} \cdot (x - 1) - (x^{2009} - 1)}{(x - 1)^2}
\]

Substitute \( x = 2 \),

\[
1 + 2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \cdots + 2008 \cdot 2^{2007} = 2009 \cdot 2^{2008} - 2^{2009} + 1 = \frac{2007}{1005} \cdot 2^{2008} + 1.
\]
10. Write the rational number $\frac{3}{8}$ as a base 7 decimal expansion

$$\frac{3}{8} = 0.a_1a_2a_3 \cdots = \frac{a_1}{7} + \frac{a_2}{7^2} + \frac{a_3}{7^3} + \cdots,$$

where $a_i \in \{0, 1, 2, 3, 4, 5, 6\}$ for $i = 1, 2, 3, \ldots$.

**Solution.**

\[
\begin{align*}
\frac{3}{8} &= \frac{a_1}{7} + \frac{a_2}{7^2} + \frac{a_3}{7^3} + \cdots \\
\frac{21}{8} &= a_1 + \frac{a_2}{7} + \frac{a_3}{7^2} + \frac{a_4}{7^3} + \cdots \\
&\quad \text{(so $a_1 = 2$. Substitute back in the equation.)}
\end{align*}
\]

\[
\begin{align*}
\frac{5}{8} &= \frac{a_2}{7} + \frac{a_3}{7^2} + \frac{a_4}{7^3} + \cdots \\
\frac{35}{8} &= a_2 + \frac{a_3}{7} + \frac{a_4}{7^2} + \frac{a_5}{7^3} + \cdots \\
&\quad \text{(so $a_2 = 4$. Substitute back in the equation...)}
\end{align*}
\]

\[
\begin{align*}
\frac{3}{8} &= \frac{a_3}{7} + \frac{a_4}{7^2} + \frac{a_5}{7^3} + \cdots.
\end{align*}
\]

where $a_i$’s begin to repeat. We can see easily that $a_1 = a_3 = a_5 = \cdots = 2$, while $a_2 = a_4 = a_6 = \cdots = 4$. So,

$$\frac{3}{8} = (0.24)_7.$$

11. Let $a$, $b$, $c$ be positive odd integers. Prove that $ax^2 + bx + c = 0$ does not have rational solutions.

**Solution.** It suffices to show that $\sqrt{b^2 - 4ac}$ is NOT a perfect square. Assume the opposite, then $b^2 - 4ac = d^2$ for some $d \in \mathbb{N}$. Since $b^2 - 4ac$ is odd, $d$ is also odd. Let $b = 2k + 1$ and $d = 2h + 1$, then

\[
\begin{align*}
4ac &= b^2 - d^2 \\
&= (b + d)(b - d) \\
&= (2k + 2h + 2)(2k - 2h) \\
&= 4(k + h + 1)(k - h) \\
ac &= (k + h + 1)(k - h).
\end{align*}
\]

Since $(k + h + 1)$ and $(k - h)$ differ by $(k + h + 1) - (k - h) = 2h + 1$, one of them is odd and the other is even. So $(k + h + 1)(k - h)$ is even, contradicting the assumption that $a$ and $c$ are both odd numbers.
12. Consider a sequence defined by \( a_1 = 1, a_{n+1} = \sqrt{2}^{a_n}, \) \( n = 1, 2, \ldots \). Does \( a_n \) converge? If yes, find is the limit \( \lim_{n \to \infty} a_n \).

\textit{Solution.}

(a) \( a_n \) is increasing. This can be easily established by induction because 1) \( a_2 > a_1 \), and 2) \( \sqrt{2}^x \) is an increasing function, \( a_{n+2} = \sqrt{2}^{a_{n+1}} > a_{n+1} = \sqrt{2}^{a_n} \) if \( a_{n+1} > a_n \).

(b) \( a_n \leq 2 \): again use induction.

Therefore \( a_n \) converges because it is a bounded increasing sequence.

(c) Let \( \alpha = \lim_{n \to \infty} a_n \). Then

\[ \alpha = \sqrt{2}^\alpha. \]

Consider the equation \( x = \sqrt{2}^x \): it is obvious that \( x = 2, 4 \) are two solutions. They are indeed the only solutions because an exponential function intersects with a line at most twice. Given that \( a_n \leq 2 \), \( \alpha = \lim_{n \to \infty} a_n = 2 \).
For each positive integer \( n > 1 \), \( A_n \) represents the area of the region restricted to the following two inequalities: 
\[
\frac{x^2}{n^2} + \frac{y^2}{n^2} \leq 1 \quad \text{and} \quad \frac{x^2}{n^2} + \frac{y^2}{n^2} \leq 1 \quad (A_n = \text{area of the intersection of the two inequalities})
\]
Find \( \lim_{n \to \infty} A_n \).

**Solution:**

There are four equalities for borders:

Ellipse 1: \( \frac{x^2}{n^2} + \frac{y^2}{n^2} = 1 \Rightarrow x^2 + n^2y^2 = n^2 \Rightarrow y = \frac{\sqrt{n^2-x^2}}{n} \) or \( y = -\frac{\sqrt{n^2-x^2}}{n} \)

Ellipse 2: \( \frac{x^2}{n^2} + \frac{y^2}{n^2} = 1 \Rightarrow n^2x^2 + y^2 = n^2 \Rightarrow y = n\sqrt{1-x^2} \) or \( y = -n\sqrt{1-x^2} \)

Also, four intersecting points:

\[
(x,y) = \left( \frac{n}{\sqrt{n^2+1}}, \frac{n}{\sqrt{n^2+1}} \right), \left( \frac{n}{\sqrt{n^2+1}}, \frac{-n}{\sqrt{n^2+1}} \right), \left( \frac{-n}{\sqrt{n^2+1}}, \frac{n}{\sqrt{n^2+1}} \right), \left( \frac{-n}{\sqrt{n^2+1}}, \frac{-n}{\sqrt{n^2+1}} \right)
\]

Notice that the region is symmetric with respect to: x-axis, y-axis, \( y=x \) and \( y=-x \).

Hence \( A_n = 8 \cdot B_n \), where \( B_n = \text{Area between the y-axis, } y=x, \text{ and } y = \frac{\sqrt{n^2-x^2}}{n} \).

As \( n \to \infty \), \( \lim_{n \to \infty} \frac{n}{\sqrt{n^2+1}} = 1 \), \( B_n \approx \text{Area of } \Delta AOB \) where \( O=(0,0), \ A=(1,0) \) and \( B=(1,1) \).

\[
\lim_{n \to \infty} A_n = 8 \cdot \left( \frac{1}{2} \cdot 1 \cdot 1 \right) = 4
\]
Problems and Solutions

1. Without direct calculation, find the limit

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{3}{n} \sqrt{1 + \frac{3i}{n}}.
\]

**Solution.** This is the Riemann sum of the integral

\[
\int_{0}^{3} \sqrt{1 + x} \, dx = \left. \frac{2}{3} (1 + x)^{3/2} \right|_{0}^{3} = \frac{14}{3}
\]

Or, consider the integral

\[
\int_{0}^{3} 3\sqrt{1 + 3x} \, dx = \frac{14}{3}
\]

2. A “code” is a sequence which consists of 5 numbers \(x_i \in \{0, 1\}\). For example, 00110 and 10010 are two different codes. The distance between two codes is the number of places where the corresponding digits are distinct. For example, distance between 00110 and 10111 is 2 because they differ in the 1st and 5th places.

(a) Find the code with maximum distance from 00101.

(b) How many codes are 2 places (distance = 2) away from 00101?

(c) Let \(d(x, y)\) represent the distance between two codes \(x\) and \(y\). Determine whether or not the inequality is true:

\[
d(x, z) \leq d(x, y) + d(y, z) \quad \text{for all } x, y, z.
\]

If true, prove it. Otherwise, give a counter example.

**Solution.**

(a) The code with maximum distance(=5) from a given code \(x\) would be the code with all places distinct from the given code. Hence the answer is 11010.

(b) \(\binom{5}{2} = 10\).
(e) Let \( x = x_1x_2x_3x_4x_5, y = y_1y_2y_3y_4y_5 \) and \( z = z_1z_2z_3z_4z_5 \). Since

\[
|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|,
\]

we take sum over \( i = 1 \) to \( 5 \),

\[
\sum_{i=1}^{5} |x_i - z_i| \leq \sum_{i=1}^{5} |x_i - y_i| + \sum_{i=1}^{5} |y_i - z_i|,
\]

i.e.

\[
d(x, z) \leq d(x, y) + d(y, z).
\]

So the triangle inequality holds.

3. A sequence \( \{a_n\} \) is defined by

\[
a_1 = \alpha, \quad a_2 = \beta, \quad a_3 = \gamma,
\]

\[
a_{n+3} = \frac{1}{3} \left( a_{n+2} + a_{n+1} + a_n \right), \quad n = 1, 2, \ldots.
\]

Does \( a_n \) converge as \( n \) approaches \( \infty \)? If yes, express the limit explicitly in terms of \( \alpha, \beta, \) and \( \gamma \).

\textit{Solution}. The characteristic equation of this recursive sequence is

\[
3r^3 - r^2 - r - 1 = 0,
\]

which has roots \( r = 1, \frac{-1 \pm i\sqrt{2}}{3} \). Since \( |\frac{-1 \pm i\sqrt{2}}{3}| \leq 1 \), \( a_n \) converges.

\[
a_4 = \frac{1}{3}a_3 + \frac{1}{3}a_2 + \frac{1}{3}a_1
\]

\[
a_5 = \frac{1}{3}a_4 + \frac{1}{3}a_3 + \frac{1}{3}a_2
\]

\[\vdots\]

\[
a_{n+3} = \frac{1}{3}a_{n+2} + \frac{1}{3}a_{n+1} + \frac{1}{3}a_n
\]

So,

\[
a_{n+3} + \frac{2}{3}a_{n+2} + \frac{1}{3}a_{n+1} = \gamma + \frac{2}{3}\beta + \frac{1}{3}\alpha.
\]

Let \( n \to \infty \), we have

\[
2 \lim_{n \to \infty} a_n = \gamma + \frac{2}{3}\beta + \frac{1}{3}\alpha.
\]
\[
\lim_{n \to \infty} a_n = \frac{1}{2} \gamma + \frac{1}{3} \beta + \frac{1}{6} \alpha.
\]

4. For each positive integer \( n > 1 \), \( A_n \) represents the area of the region restricted to the following two inequalities: \( \frac{x^2}{n^2} + y^2 = 1 \) and \( \frac{y^2}{n^2} + x^2 = 1 \). Find \( \lim_{n \to \infty} A_n \).

**Solution.** See Appendix A.

5. Let \( \alpha_n = \left( a + \sqrt{a^2 - 2a + 2} \right)^n \) for some integer \( a > 1 \). Show that \( \lim_{n \to \infty} \left( \alpha_n - \lfloor \alpha_n \rfloor \right) = 1 \).

**Solution.** Let \( \beta_n = \left( a - \sqrt{a^2 - 2a + 2} \right)^n \). Then

\[
\alpha_n + \beta_n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} (\sqrt{a^2 - 2a + 2})^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} (- \sqrt{a^2 - 2a + 2})^k
\]

which is an integer. Since

\[
a - \sqrt{a^2 - 2a + 2} < a - \sqrt{a^2 - 2a + 1} = a - (a - 1) = 1,
\]

we have

\[
\beta_n = \left( a - \sqrt{a^2 - 2a + 2} \right)^n \to 0 \text{ as } n \to \infty.
\]

So, \( \lfloor \alpha_n \rfloor = \alpha_n - \beta_n - 1 \). Let \( n \) approach \( \infty \), and we have \( \lim_{n \to \infty} \left( \alpha_n - \lfloor \alpha_n \rfloor \right) = 1 \).

6. Show that

\[
\cos 1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots + \frac{(-1)^k}{(2k+1)!} + \cdots
\]

is an irrational number.

**Solution.** Assume the opposite, i.e. \( \cos 1 = \frac{p}{q} \), \( p, q \in \mathbb{N} \), \( (p, q) = 1 \). Then

\[
\frac{p}{q} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots + \frac{(-1)^k}{(2k+1)!} + \cdots
\]

Choose \( k \) so that \( 2k + 1 > q \). Then \( q \) divides \( (2k+1)! \). Then

\[
\left| \frac{p}{q} - \left( 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots + \frac{(-1)^k}{(2k+1)!} \right) \right| = \left| \frac{(-1)^{k+1}}{(2k+3)!} + \cdots \right| < \frac{1}{(2k+3)!}
\]
However

\[
\left| \frac{p}{q} - \left( 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots + \frac{(-1)^k}{(2k+1)!} \right) \right|
\]

is a nonzero fraction of denominator \((2k + 1)!\), and is at least \(1/(2k + 1)!\). This is a contradiction.