Individual: 1. The decimal representation is 0.142857. 2006 modulo the period of 6 is 2. So the 2006th digit is \([4]\).

2. Let \(1 \leq n \leq 1000\) and \(n = x^2 - y^2 = (x - y)(x + y)\). Since \(x - y, x + y\) are both odd or both even, \(n\) is the difference of two squares if and only if \(n\) can be factored as the product of two integers with the same parity. The only \(n\) that cannot be factored in this manner are integers of the form \(2k\), where \(k\) is odd. Since \(1 \leq k \leq 500\), there are 250 such \(k\). Hence there are \(1000 - 250 = \frac{750}{3}\) integers \(n\) with the requested property.

3. There are 10 equiprobable ways to choose 3 people from a group of 5 people. 3 of these involve choosing both women. So the probability is \(\frac{3}{10}\).

4. Detailed proof \([24 = 2 \times 3 \times 4]\).

5. Let \(x\) and \(y\) be the respective two-digit and three-digit number. We are given the equation \(1000x + y = 9xy\). Now \(y(9x - 1) = 1000x\), so \(x\) divides \(y(9x - 1)\). Since \(x\) and \(9x - 1\) have no factors in common, \(x\) divides \(y\). Writing \(y = xk\), the equation becomes \(1000 = k(9x - 1)\). Hence \(k\) and \(9x - 1\) are factors of 1000. Since \(x\) is a two-digit number, \(98 \leq 9x - 1 \leq 999\), and \(9x - 1\) must then equal 100, 125, 200, 250, 500. Hence \(9x - 1 = 125\), \(x = 14\), \(k = 8\), and \(y = 112\). Then \(x + y = 126\).

6. Assume that the statement is false. Let \(n\) be the smallest natural number such that \(L_n\) and \(L_{n+2}\) have a nontrivial factor \(s\). Since \(L_2 = 2+7 = 9\), \(n > 0\). \(L_{n+2} = L_n + L_{n+1}\) implies that \(s\) divides \(L_{n+1}\). \(L_{n+1} = L_{n-1} + L_n\) implies that \(s\) divides \(L_{n-1}\). This contradicts the minimality of \(n\).

7. Let \(S\) be the set of all positive integers \(n\) such that the equation \(7a + 11b = n\) does not have a solution with nonnegative integers \(a, b\). We are looking for the largest element of \(S\). Let \(r\) be the remainder of \(n\) when divided by 7. If \(r = 0\), then \(n = 7a\), so \(n\) is not an element of \(S\). If \(r = 1\), then \(1, 8, 15 \in S\). Now \(22 = 2(11)\), so every other element \(n\) with \(r = 2\) is not in \(S\) as \(22 + 7a = 7a + 11(2)\). Similarly, if \(r = 2\), then \(2, 9, 16, 23, 30, 37 \in S\), but \(44 + 7a = 7a + 11(4)\), so no 37 is the largest element of \(S\) with \(r = 2\).

When \(r = 3\), the largest \(n \in S\) is 59; when \(r = 4\), the largest \(n \in S\) is 4; when \(r = 5\), the largest \(n \in S\) is 26; and when \(r = 6\), the largest \(n \in S\) is 48. Hence the largest element in \(S\) is \(59\).

8. \((\frac{1}{3})^{5/16}\)

9. Factor \(10^5\) into primes, \(10^5 = 2^5 \cdot 5^5\). Hence \(10^5\) has 36 distinct positive divisors, each of the form

\[2^i \cdot 5^j \quad 0 \leq i, j \leq 5\]

Fix an \(i\), we get divisors \(2^5 5^1, 2^5 5^2, 2^5 5^3, 2^5 5^4, 2^5 5^5\). The product of the 6 divisors above is \(2^6 \times 15.5^6 \times 15 = 10^{90}\). Let \(i = 0, 1, ..., 5\), we get

\[N = 2^6 \times 15 \cdot 5^6 \times 15 = 10^{90} \quad \Rightarrow \quad \log(N) = \log(10^{90}) = 90\]

10. From \(f(f(x)) = x\) for \(x \neq -d/c\), one can deduce that \(f\) is one-to-one as \(f(a) = f(b)\) implies \(a = f(f(a)) = f(f(b)) = b\). Inspection suggests that
f sends $R - \{-d/c\}$ to $R - \{a/c\}$ as $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = a/c$. By examination, the equation $f(x) = a/c$ has either no solution or infinitely many solutions. As $f$ is one-to-one, $a/c$ cannot be in the range. Let $y = a/c$.

We now show that $y = -d/c$. Assume that $y \neq -d/c$. Then $y$ is in the domain of $f$ and $z = f(y) = y$. If $f(z) = f(f(y)) = y$ and $y$ would be in the range of $f$, which gives a contradiction. If $z = -d/c$, then the expression $f(f(y))$ is not defined, contradicting the hypothesis in the problem. Hence $y = -d/c$ and $a/c = -d/c$.

From $f(19) = 19$, one obtains the equation $19^2 c + 19(d - a) = b$. From $f(97) = 97$, one obtains $97^2 c + 97(d - a) = b$. Equating these, one obtains $a/c - d/c = 116$. Since $a/c = -d/c$, we have $y = a/c = \frac{58}{2}$.

11. Note that $x^2 + y^2 - 2xy = (x - y)^2 \geq 0$, and $x^2 + y^2 + 2xy = (x + y)^2 \geq 0$. Thus $\frac{1}{2}(x^2 + y^2) \geq xy \geq \frac{-a}{2}(x^2 + y^2)$.

$$\frac{a}{2}(x^2 + y^2) \geq axy \geq \frac{-a}{2}(x^2 + y^2) \text{ if } a \geq 0$$

$$\frac{-a}{2}(x^2 + y^2) \geq axy \geq \frac{a}{2}(x^2 + y^2) \text{ if } a < 0$$

For any solution $(x, y)$ of $1 = 2x^2 + axy + 2y^2 = 2(x^2 + y^2) + axy$, we have:

$$1 \geq 2(x^2 + y^2) - \left(\frac{a}{2}\right)(x^2 + y^2) = \left(4 - \frac{a}{2}\right)(x^2 + y^2) \text{ if } a \geq 0$$

$$1 \geq 2(x^2 + y^2) + \left(\frac{a}{2}\right)(x^2 + y^2) = \left(4 + \frac{a}{2}\right)(x^2 + y^2) \text{ if } a < 0$$

In the first case, in order for $x^2 + y^2 \leq 1$, $4 \geq a \geq 2$.

Similarly, when $a < 0$, we need $-2 \leq a < 0$ (so that $x^2 + y^2 \leq 1$). Thus the minimum value of $a$ is $-2$.

12. Rewriting the limit we get $\lim_{x\to\infty} \int_{e^{x^2}}^{e^{x^2 - 1}} a^2 dt$. By l'Hôpital's rule this is equal to $\lim_{x\to\infty} \frac{e^{x^2}}{2e^{x^2} - e^{x^2 - 1}}$. Simplifying we get $\lim_{x\to\infty} \frac{x^2}{2x^2 - 1} = \frac{1}{2}$. 


Team:
1. The angle $\angle A_nA_1A_2$ has measure $180 - 360/n$ degrees. Hence the angle $\angle A_nA_1B$ will have measure $120 + 360/n$ degrees. If $A_n, A_1, B$ are three consecutive vertices of a regular $m$-gon, then $120 + 360/n = 180 - 360/m$ for some $m$. Then $mn - 6n - 6m = 0$. Adding 36 to both sides, $(m-6)(n-6) = 36$ and $n - 6$ is a divisor of 36. The largest possible value for $n$ is $42$, with a corresponding value of $m = 7$.
2. Detailed proof.
3. $0$
4. According to the assumption, write $(a_{n+1} \ b_{n+1}) = 2A \cdot (a_n \ b_n)$
   where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
   Thus
   $(a_{n+3} \ b_{n+3}) = 2A \cdot (a_{n+2} \ b_{n+2}) = 4A^2 \cdot (a_{n+1} \ b_{n+1}) = 8A^3 \cdot (a_n \ b_n) = \begin{pmatrix} 8 & 24 \\ 0 & 8 \end{pmatrix} \cdot (a_n \ b_n)$
   Therefore
   $x = 8, y = 24, z = 0, w = 8$
5. Consider the point $P = (n, \sqrt{n^2 + 1})$ on the curve $y^2 - x^2 = 1$. The vertical line $x = n$ and $y = x$ intersect at the point $Q = (n, n)$.
   By similarity $\triangle ONQ \sim \triangle PMQ$, we have
   $d_n = \frac{\sqrt{1 + n^2 - n}}{\sqrt{2n}} \Rightarrow d_n = \frac{(\sqrt{1 + n^2 - n})(\sqrt{1 + n^2} + n)}{\sqrt{2(1 + n^2) + n}}$
   Hence
   $n \cdot d_n = \frac{n}{\sqrt{2(1 + n^2 + n)}} = \frac{1}{\sqrt{2(\frac{1}{n^2} + 1 + 1)}}$
   So,
   $\lim_{n \to \infty} n d_n = \lim_{n \to \infty} \frac{1}{\sqrt{2(\frac{1}{n^2} + 1 + 1)}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$
6. Let $\alpha_n = (2 + \sqrt{3})^n$ and $\beta_n = (2 - \sqrt{3})^n$. By Binomial Theorem (or by Mathematical Induction), $\alpha + \beta \in \mathbb{N}$. Since $0 < \beta_n < 1$, we have
   $[\alpha_n] = \alpha_n + \beta_n - 1$,
   therefore
   $\alpha_n - [\alpha_n] = 1 - \beta_n$.
   Since $\lim_{n \to \infty} \beta_n = 0$,
   $\lim_{n \to \infty} (\alpha_n - [\alpha_n]) = \lim_{n \to \infty} (1 - \beta_n) = 1$. 
2. a. Consider the function \( f : A \to O = \{ \text{positive odd integers <2006} \} \) defined by
\( f(a) = b \), where \( a = 2^ib \) and \( b \) is odd. There are 1003 odd integers in \( O \), so by the pigeonhole
principal there must exist \( a_1 \) and \( a_2 \) such that \( f(a_1) = f(a_2) \), so \( a_1 \) divides \( a_2 \) or
vice versa.

b. No element of \( C = \{ a : 1004 \leq a \leq 2006 \} \) is a multiple of another.

3. The answer is 0. If \( f(x) = \sqrt{1-x^2} \) then \( \sqrt{1-x^2} = f^{-1}(x) \). Both graphs start at \((0,1)\)
and decrease to end at \((0,1)\) and intersect at a point \((A,A)\) (How would you find the
numerical value of this point?) The region that is between the graphs is symmetric about
the line \( y = x \). Any portion of this region that is above \( c < x < d < A \) and for which
\( f(x) > f^{-1}(x) \) for all \( c < x < d \) will be reflected, via reflection in the line \( y = x \), into a
region of equal area for which \( f(x) < f^{-1}(x) \), and vice versa. (Actually, \( f(x) > f^{-1}(x) \)
for \( x < A \). How could your find this out without a calculator?)

4. Well, \( p^2 - 1 = (p+1)(p-1) \), so one of these factors is divisible by 3, since \( p \) is not.
And \( p = 4k - 1 \) or \( 4k - 3 \) so one of these factors is divisible by 4. The other is divisible by
2.

8. The limit exists because the sequence is decreasing and bounded below. Denoting the
desired number by \( N \),
\[ \ln N = \ln 3 \left( \frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \cdots \right) = -\ln 3 \sum_{k=1}^{m} \frac{k}{5^k} \]
\[
\text{Let } f(x) = \sum_{k=1}^{m} \frac{(x/5)^k}{k}.
\]
Then \( f'(x) = \sum_{k=1}^{m} \frac{kx^{k-1}}{5^k} \). Both are convergent power series for \( |x| < 5 \), with \( f(x) \) a
geometric series converging to \( (x/5) \frac{1}{1-x/5} \).

Then \( f'(x) = \frac{1}{5} \frac{1}{(1-x/5)} + \frac{x}{5} \left( \frac{1}{1-x/5} \right)^2 \). Therefore
\[ \ln N = (-\ln 3) f'(1) = (-\ln 3) ((1/5)(5/4) + (1/25)(25/16)) = -\ln 3 \cdot 5/16 \]
So \( N = (1/3)^{5/16} \)