1. \( \frac{1}{5} = 20\% \) There are 5 possible rolls of the dice whose sum is 8: (2, 6), (3, 5), (4, 4), (5, 3), (6, 2). Of these 5, one is the possibility (4, 4). So the probability is 1/5.

2. \( [120] \) There are 2 possibilities for the way the first two letters can be vowels. The remaining 5 letters can be rearranged in \( 5! \) possibilities (the doubled T means each of 5! possibilities are counted twice). Total, we have \( 2 \times 5! = 120 \) rearrangements.

3. \( \frac{8}{3\sqrt{3}} = \frac{8\sqrt{3}}{9} \) Let \( s \) be the length of an edge of the cube. Then the diagonal of a side has length \( \sqrt{2}s \) and the diagonal of the cube is \( \sqrt{3}s \). Now since the cube is inscribed in the sphere, \( \sqrt{3}s = 2 \) and \( s = 2/\sqrt{3} \), so the volume is \( s^3 = 8/(3\sqrt{3}) \).

4. \( [14] \) We want \( 2^{-n} < 10^{-4} \), so \( 2^n > 10^4 = 10000 \). Now \( 2^8 = 256, 2^{10} = 1024, 2^{13} = 8192 \), so \( n = 14 \) is the smallest solution.

5. \( [6\sqrt{6}] \) Let \( \triangle ABC \) be the triangle and \( AB = 5, BC = 6, AC = 7 \). Let the altitude from \( B \) to \( AC \) intersect \( AC \) at \( D \). Let \( x \) be the length \( AD \). Then \( 7 - x \) is the length of \( DC \). Now by the Pythagorean theorem, \( (BD)^2 + x^2 = 5^2 \), and \( (BD)^2 + (7 - x)^2 = 6^2 \). Then \( 25 - x^2 = 36 - (7 - x)^2 = -13 + 14x - x^2 \). So \( 14x = 38 \) and \( x = \frac{19}{7} \). Then the length of \( BD \) is \( y\sqrt{25 - (19/7)^2} = 12\sqrt{6}/7 \). So the area of the triangle is \( 7y/2 = 6\sqrt{6} \).

6. \( [12] \) The fourth derivative of \( \cos(x) \) is \( \cos(x) \), so \( n \) needs to be a multiple of 4. Correspondingly, we need \( n \) to be a multiple of 3 from the hypothesis on \( f(x) \). \( n = 12 \) is the smallest number that satisfies both conditions.

7. \( [4] \) The remainder of \( 0! + 1! + 2! + \cdots + 1000! \) when divided by 5 is \( 1 + 1 + 2 + 1 + 4 + 0 + 0 + \cdots + 0 = 9 = 4 \). Cubed, the remainder is 4.

8. \( \sqrt{a} \) From the limit, we have \( A = \lim_{x \to 0} x + \frac{a}{x} \). So \( A = a/A \) and \( A^2 = a \). So \( A = \sqrt{a} \).

9. \( (a) \ 23 (b) \ 12 \) To maximize the sum, you want the smallest numbers 1, 2 to be \( a_1, a_4 \) and be on the edges of the sum. To minimize the sum, you want the largest numbers on the edges. Trial and error shows that \( 2(4) + (4)(3) + (3)(1) = 23 \) and \( 4(1) + (1)(2) + (2)(3) = 12 \).

10. \( [57] \) Let \( r, w, b \) be the number of red, white, and blue marbles. Then \( b \leq w/2, b \leq r/3, \) and \( w + b \geq 55 \). Since \( 2b \geq w, 3b \geq 55, b \geq 19 \). Minimizing \( w, r \), we get \( w = 36, \) and \( r = 57 \).

11. \( [162] \) We have \( a_3 = a_1 + a_2, a_4 = a_1 + 2a_2, a_5 = 2a_1 + 3a_2, a_6 = 3a_1 + 5a_2, a_7 = 5a_1 + 8a_2 \). Now \( 5a_1 + 8a_2 = 100 \) has only one integral solution with \( a_2 > a_1 > 0 \). It is \( a_1 = 4, a_2 = 10 \). Then the sequence is 4, 10, 14, 24, 38, 62, 100, 162, \ldots .

12. \( (500, 502), (164, 170) \) We have \( 2004 = n^2 - m^2 = (n - m)(n + m) \). Since \( 2004 = 2^2(3)(167), \) and \( n - m, n + m \) have the same parity, we must have that \( n - m, n + m \) are even. Since \( n + m > n - m \), there are only two possibilities: \( (n + m, n - m) = (1002, 2), (334, 6) \). Solving for \( (m, n) \), we obtain \( (500, 502), (164, 170) \) as solutions.
13. \( n = 1, 2, 5, 125 \) If \( 1/n \) has a terminating decimal expansion, then \( n = 2^k 5^l \). So its last digit is 0, 1, 2, 4, 5, or 8. Now 1 can only occur when \( n = 1 \). So the last digits of \((n, n + 3)\) could be (2, 5), (5, 8), or (1, 4). Since \( n < 1000 \), the first case can only happen when \((n, n + 3) = (2, 5)\), the second case can only happen when \((n, n + 3) = (5, 8), (125, 128)\), and the third case can only happen when \((n, n + 3) = (1, 4)\). So \( n = 1, 2, 5, 125 \).

14. radius = \( \sqrt[3]{\frac{V}{2\pi}} \), height = \( \sqrt[3]{\frac{4V}{\pi}} \) Let \( r, h \), be the radius and height of the cylinder.

Then \( V = \pi r^2 h \). Now the surface area is \( S = 2\pi rh + 2\pi r^2 \). Substituting \( h = V/(\pi r^2) \), we have \( S(r) = 2V/r + 2\pi r^2 \). Differentiating, \( S'(r) = -2V/r^2 + 4\pi r \). \( S'(r) = 0 \) gives \( r^3 = V/(2\pi) \), so \( r = \sqrt[3]{V/(2\pi)} \). So \( h = \sqrt[3]{4V/\pi} \).
1. False. Consider the following situation. Suppose Player A gets one hit in her one at-bat in the first half of the season, and gets 1 hits in 199 at-bats in the second half of the season. Then Player A bats 1.000 in the first half of the season, and .005 in the second half of the season, and .0100 for the entire season.

Now if Player B gets 100 hits in 199 at-bats in the first half of the season and no hits in 1 at-bat the second half of the season. Then Player B bats .505 in the first half of the season, .000 in the second half of the season, and .500 for the entire season.

2. \(A = 20\). We want to solve the following equations:

\[
\begin{align*}
30 + B + C &= 4A \\
60 + A + D &= 4B \\
70 + B + C &= 4D \\
40 + A + D &= 4C
\end{align*}
\]

Solving this equations gives \(A = 20, B = 35/2, C = 45/2, D = 30\).

3. \(\det(A) = (a - b)^{n-1}[a + (n - 1)b]\). Let \(A_n\) denote the \(n \times n\) matrix. We can prove this by induction. When \(n = 1\), we have \(\det(A_1) = a\). We assume that the formula is proved for a \(n \times n\) matrix. Given \(A_{n+1}\) subtract the top row from the bottom row. Then the bottom row has \(b - a\) in the first column, and \(a - b\) in the last column. Expanding along the last row, we have

\[
\begin{vmatrix}
0 & 0 & \ldots & 0 & b \\
0 & a - b & \ldots & 0 & b \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & a - b & b \\
\end{vmatrix}
\]

Now

\[
\begin{vmatrix}
b & b & \ldots & b & b \\
a & b & \ldots & b & b \\
b & a & \ldots & b & b \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
b & b & \ldots & a & b \\
\end{vmatrix} = (-1)^{2n+2}(a - b)|A_n|.
\]

And

\[
|A_{n+1}| = b(a - b)(a - b)^{n-1} + (a - b)((a - b)^{n-1}[a + (n - 1)b] = (a - b)^n[b + a + (n - 1)b] = (a - b)^n[a + nb].
\]

4. (a) The set \(S = \{0, 1, 2, 3, \ldots, 9\}\) satisfies the hypothesis.

(b) The statement is true. Proof: Let \(n\) be the number of elements of \(A\). The hypothesis on \(A\) implies \(n > \frac{2}{3}|S|\). Let \(m\) be the number of elements of \(S\) not in \(A\). Then \(m < |S|/3\) and \(2m < n\). Assume that there is an element \(x \in S\) that is not the sum or difference of two elements of \(A\). Then if \(y \in A\) and \(y < x\), then \(x = y + (x - y)\), so \(x - y \in S - A\) (otherwise \(x\) is the sum of two elements of \(S\)). Similarly if \(y \in A\) and \(y > x\), then \(x = y - (y - x)\), so \(y - x \in S - A\) (otherwise \(x\) is the difference of two elements of \(S\)). Thus if \(y_1, y_2, \ldots, y_n\) are the elements of \(A\), then \(x - y_1, |x - y_2|, \ldots, |x - y_n|\), is a list of elements of \(S - A\). As \(|x - y_i|\) may equal \(|y_j - x|\), the second list may list some elements of \(S - A\) twice. But
there are at least \((n - 1)/2\) + 1 distinct elements. Hence \(m \geq [(n - 1)/2] + 1 \geq n/2\) and \(2m \geq n\). But this contradicts the inequality \(2m < n\) derived above. Hence our assumption is incorrect and every element of \(S\) must be the sum or difference of two elements of \(A\).

5. Both associativity and commutativity hold.

(b) Commutativity:

\[
\begin{align*}
    a \circ b &= \left( e \circ a \right) \circ \left( e \circ b \right) \\
    &= \left( e \circ b \right) \circ \left( e \circ a \right) \quad \text{by the given} \\
    &= b \circ a.
\end{align*}
\]

(a) Associativity:

\[
\begin{align*}
    a \circ (b \circ c) &= a \circ (c \circ b) \quad \text{by (b)} \\
    &= \left( a \circ e \right) \circ (c \circ b) \\
    &= (a \circ b) \circ (e \circ c) \\
    &= (a \circ b) \circ c
\end{align*}
\]

6. If \(n = 1\), \(\phi(1) = \gamma(1) = 1\). Now \(\phi(2) \neq \gamma(2)\), and for \(n > 2\), \(\phi(n)\) is even. Now if \(\gamma(n)\) is even, \(n\) is even and \(\gamma(n) = 2^2m\), with \(m\) odd. Rewriting the formula for \(\phi(n)\) as \(\prod p_i^{k_i-1}(p_i - 1)\), where \(n = \prod p_i^{k_i}\) is the prime factorization of \(n\), we see that there can only be two odd primes \(p\) in the factorization. For if there are more than 2 odd primes \(p\), \(\phi(n)\) is divisible by 8 and the equality \(\phi(n) = \gamma(n)^2\) cannot hold. Hence there are three cases:

- 1) \(n = 2^k\),
- 2) \(n = 2^kp^l\),
- 3) \(n = 2^kp^lq^m\),

for which \(\phi(n) = \gamma(n)^2\) holds. In (1), \(2^{k-1} = 4\), so \(k = 3\), and \(n = 8\). In (2), \(2^{k-1}p^l-1(p-1) = 4p^2\). So \(l = 3\), and \(2^{k-1}(p - 1) = 4\). If \(p - 1 = 2\), \(p = 3\) and \(k = 2\), so \(n = 2^2\times 3 = 108\). If \(p = 4\), \(p = 5\) and \(k = 1\) so \(n = 2^4\times 5^3 = 250\). In case (3), \(2^{k-1}(p-1)q^{l-1}(q-1)q^{m-1} = 4p^2q^2\). We can assume \(q\) is the largest prime. Then \(m = 3\) and we have \(2^{k-1}(p-1)q^{l-1}(q-1) = 4p^2q^2\). Since \(p - 1\), \(q - 1\) are even \(k = 1\). Since \((p - 1, p) = 1\), \(p - 1\) divides 4. Now \(p - 1 = 1\) forces \(p = 2\) and \(p - 1 = 4\) forces \(q = 2\). So the only choice is \(p - 1 = 2\). Then \(p = 3\) and \(3^{l-1}(q - 1) = 18\). Then \((l, q) = (1, 18), (2, 7)\) are the only two solutions. These lead to \(n = 2(3)(19)^3 = 41154\) and \(n = 2(3)^2(7)^3 = 6174\). So the six values of \(n\) are

\[
\begin{align*}
    n &\text{= 1, 8, } 2^3, 108 = 2^23^3, 250 = 2(5)^3, \\
    6174 = 2(3)^2(7)^3, 41154 = 2(3)(19)^3
\end{align*}
\]

7. For \(n = 2, 3\), we have \(\cos(\pi/2n) = \sqrt{n}/2\). Then using the half-angle identity,

\[
\cos(\theta) = \sqrt{\frac{\cos(2\theta) + 1}{2}}, \quad \text{for } 0 \leq \theta < \pi/2,
\]

we have \(\cos(\pi/(4n)) = \frac{1}{2} \sqrt{2 + \sqrt{n}}\). In general,

\[
\cos(\pi/(2^k n)) = \frac{1}{2} \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{n}}}},
\]
where there are $k$ square roots. We can prove this by induction. We have already established it for $k = 2$. Then assuming the identity is true for $k$, we have

$$
\cos(\pi/(2^{k+1}n)) = \sqrt{\frac{1 + \cos(\pi/(2^k n))}{2}}
$$

$$
= \sqrt{\frac{1}{4}(2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{n}}}})}
$$

$$
= \frac{1}{2}\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{n}}},}
$$

where the last equation has $k + 1$ square roots. Now $\sin(\theta) = \sqrt{\frac{1 - \cos(2\theta)}{2}}$, so

$$
R_k(n) = \sqrt{2 - 2\cos(\pi/(2^k n))} = 2\sin(\pi/(2^{k+1}n)), \text{ for } n = 2, 3.
$$

Then

$$
\lim_{k \to \infty} \frac{R_k(2)}{R_k(3)} = \lim_{k \to \infty} \frac{2\sin(\pi/(2^{k+1}2))}{2\sin(\pi/(2^{k+1}3))} = \lim_{x \to 0} \frac{\sin(x/2)}{x/2}\frac{x/3}{\sin(x/3)}
$$

$$
= \lim_{x \to 0} \frac{\sin(x/2)}{x/2} \lim_{x \to 0} \frac{x/3}{\sin(x/3)} \lim_{x \to 0} \frac{3}{2}
$$

$$
= 1 \cdot 1 \cdot \frac{3}{2} = \frac{3}{2}.
$$